UNIVERSIDAD AUTÓNOMA DE MADRID

TESIS DOCTORAL

Anomalous Transport in Hydrodynamics and Gauge/Gravity Duality Out of Equilibrium

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Many things can happen in four years, but none of them has as much of an impact on you as the people you meet along the way.

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Summary

Anomalous transport has been a topic of great interest in the last decade. New dissipationless contributions to the charge and energy currents arise due to the existence of a chiral anomaly, which is the breaking at the quantum level of the chiral symmetry.

Those transport phenomena are studied using holography because they are expected to be important in experimental setups at strong coupling. Moreover, we know that the anomaly has the same value independently of the energy scale, so we can expect to extract the universal behavior of anomalous transport by performing our computations in the nonperturbative regime. Therefore, in this thesis we use the AdS/CFT correspondence to study different features of anomaly induced transport. Let us now summarize the content of each chapter.

In Chapter 1, anomalous transport is presented in relation to the experimental setups in which it appears and the content of the other chapters is summarized. In Chapter 2, all the theoretical background required for this thesis is introduced. In particular, the chapter includes sections for anomalies, relativistic hydrodynamics, linear response theory and holography. Both Chapters 1 and 2 serve as an introduction.

In Chapter 3, we explore how symmetries in gravity can be exploited to simplify the computation of anomalous transport coefficients. In holography, the radial coordinate of AdS is dual to an energy scale for the field theory. Therefore, holographic renormalization group flows can be better understood through radially conserved charges. It has been shown in previous works that the bulk gauge and diffeomorphism symmetries can be used to find conserved quantities associated to the charge and heat currents. However, the application to anomalous transport is cumbersome, because Chern-Simons terms in the action break gauge invariance and, moreover, the gravitational Chern-Simons involves a higher number of derivatives. Nevertheless, we show that conservation laws can still be derived and they allow to obtain the field theory observables in terms of quantities evaluated at the horizon.

In Chapter 4, we study a case in which it becomes evident that the construction of membrane currents from Chapter 3 is necessary to compute the charge and energy currents. Translation symmetry breaking is introduced through the use of spatially linear scalar field backgrounds and a further coupling between the scalars and an electromagnetic field is included to produce disorder on the charged sector. The result we find is that the electric conductivity can vanish for certain values of the translation symmetry breaking couplings, and the chiral magnetic and chiral vortical conductivities are not affected by the inclusion of momentum relaxation and charged disorder. This is to be expected from the dissipationless nature of anomalous transport.

In Chapter 5, we study the holographic model of Weyl semimetals that was proposed by our research group some months before the beginning of this PhD. Its

main feature is the appearance of a topological quantum phase transition between a topological phase with nonvanishing anomalous Hall effect and a trivial phase without such effect. Our work has focused on studying the universality of this phase transition as a function of the free parameters of the model and then extending the computation of the anomalous Hall effect in the vector current to the analogous contribution in the axial current. As a result, we find that the phase transition appears for a large region of parameter space, and the axial Hall effect can be found to give the expected value once a renormalization of the external axial fields due to their coupling to the scalars is considered.

In Chapter 6, we extend the study of anomalous transport to out of equilibrium setups. In order to do that, we use generalized Vaidya metrics with momentum relaxation to induce sudden changes in the energy and charge of the system. As a result, we find that the chiral magnetic effect presents a significantly large equilibration time that depends on the length of the Vaidya quench and the momentum relaxation parameter. We expect this to give some hints on the phenomenology of the quark-gluon plasma produced in heavy ion collisions. In particular, we propose an explanation of why the experimental results from the RHIC collider in Brookhaven seem to be more compatible with anomalous transport than the ones from the LHC in Geneva.

In Chapter 7, a discussion in English and Spanish of the work done and the results obtained in this thesis is included. Finally, a list of all the references cited in the chapters closes the thesis.

Resumen

El transporte anómalo ha despertado gran interés en la última década. Nuevas contribuciones no disipativas aparecen en las corrientes de carga y energía como consecuencia de la existencia de una anomalía quiral, que es la rotura a nivel cuántico de la simetría quiral.

La holografía permite estudiar estos fenómenos de transporte porque se espera que sean importantes en sistemas experimentales en acoplo fuerte. Además, se sabe que la anomalía tiene el mismo valor independientemente de la escala de energías, así que podemos extraer el comportamiento universal del transporte anómalo a través de cálculos en el régimen no perturbativo. Por tanto, en esta tesis usamos la correspondencia AdS/CFT para estudiar diferentes aspectos del transporte inducido por anomalías. A continuación resumiremos el contenido de cada capítulo.

En el Capítulo 1, presentamos el transporte anómalo en relación a los sistemas experimentales en los que aparece y resumimos el contenido del resto de capítulos. En el Capítulo 2, se introduce todo el aparataje teórico necesario para esta tesis. El capítulo posee secciones sobre anomalías, sobre hidrodinámica anómala, sobre teoría de respuesta lineal y sobre holografía. Ambos Capítulos 1 y 2 sirven de introducción.

En el Capítulo 3, exploramos cómo las simetrías en gravedad pueden ser usadas para simplificar el cálculo de los coeficientes de transporte anómalo. En holografía, la coordenada radial de AdS es dual a una escala de energía de la teoría de campos. Por tanto, los flujos del grupo de renormalización holográfico se pueden entender mejor utilizando cargas conservadas radialmente. En trabajos previos se ha mostrado que las simetrías gauge y de difeomorfismos del bulk permiten encontrar cantidades conservadas asociadas a las corrientes de carga y calor. Sin embargo, la aplicación de estos resultados a transporte anómalo es completa porque los términos de Chern-Simons de la acción rompen la invariancia gauge y, además, el término Chern-Simons gravitacional incluye un número más alto de derivadas. Pese a estas dificultades, mostramos que se pueden derivar leyes de conservación y estas permiten la obtención de los observables de la teoría de campos en términos de cantidades evaluadas en el horizonte.

En el Capítulo 4, estudiamos un caso en el que se hace evidente la necesidad de utilizar las corrientes de membrana del Capítulo 3 para calcular las corrientes de carga y energía. La simetría translacional está rota a causa de que los campos escalares presenten un perfil lineal en las coordenadas espaciales, y otro acoplo entre los escalares y el campo electromágnetico es responsable de producir desorden en el sector cargado. Como resultado de la no conservación del momento y de la inclusión de desorden, la conductividad eléctrica se anula para ciertos valores de los parámetros de rotura de simetría translaccional, mientras que las conductividades quiral magnética y quiral vorticial no cambian. Esto último es esperable a partir de la naturaleza no disipativa del transporte anómalo.

En el Capítulo 5, estudiamos el modelo holográfico de semimetales de Weyl que fue propuesto por nuestro grupo de investigación algunos meses antes del inicio del doctorado. Su principal ingrediente es la aparición de una transición de fase cuántica entre una fase topológica con efecto Hall anómalo no nulo y otra fase trivial que no presenta tal efecto. Nuestro trabajo se centra en el estudio de la universalidad de la transición de fase, teniendo en cuenta su dependencia de los parámetros libres del modelo, y en la extensión del cálculo del efecto Hall anómalo en la corriente vectorial a la contribución análoga en la corriente axial. Como resultado, encontramos que la transición de fase aparece para una gran región del espacio de parámetros y que el efecto Hall axial tiene el valor esperado una vez consideramos la renormalización de los campos axiales externos por su acoplo a los escalares.

En el Capítulo 6, extendemos el estudio de transporte anómalo a sistemas fuera del equilibrio. Con este objetivo, utilizamos métricas de Vaidya generalizadas con la conservación del momento rota para producir cambios repentinos en la energía y carga del sistema. Como resultado, encontramos que el efecto quiral magnético presenta tiempos de equilibración largos que dependen de la duración de los *quenches* y del parámetro de relajación del momento. Creemos que esto puede aportar pistas sobre la fenomenología del plasma de quarks y gluones producido en las colisiones de iones pesados. En particular, proponemos una explicación de por qué los resultados experimentales del colisionador RHIC de Brookhaven parecen más compatibles con la existencia de transporte anómalo que los resultados del LHC de Ginebra.

En el Capítulo 7, se incluye una discusión en inglés y castellano del trabajo hecho y de los resultados obtenidos en esta tesis. Por último, una lista de todas las referencias citadas en los capítulos cierra la tesis.

Contents

Chapter 1

Introduction

Anomalies represent the breaking at the quantum level of a classical symmetry. Exactly half a century has passed since they were first found. Anomalies are known to be an essential feature of the quantum field theory formalism and it is thus necessary to study them properly in order to grasp the physics of a model. However, only recently they have been understood as the source of quantum transport (see [94, 102] for reviews).

According to this, anomalies produce macroscopic effects in systems with an imbalance of chiral fermions. Anomalous transport phenomena represent new contributions to charge and energy currents proportional to magnetic field and vorticity, and one of their most interesting properties is that they are dissipationless.

The fact that they are nondissipative means that they do not contribute to the entropy production and they do not alter the capability of the system to produce work. Since work can be computed as the dot product of force times displacement, this can be seen in simple terms from the fact that both magnetic field and vorticity are orthogonal to the force they produce. More rigorously, their dissipationless nature can be seen from the fact that the transport coefficients are even under time reversal and, therefore, they only appear on the hermitian part of the response function from linear response theory. We will come back at this when we review linear response in Section 2.3.

The precise form of the transport coefficients is given by the following expressions

$$\begin{aligned}
\vec{J} &= \frac{e^2\mu_5}{2\pi^2}\vec{B} + \frac{e\mu\mu_5}{\pi^2}\vec{\omega}\,, \\
\vec{J}_\epsilon &= \frac{e\mu\mu_5}{2\pi^2}\vec{B} + \left(\frac{\mu^2\mu_5}{\pi^2} + \frac{\mu_5^3}{3\pi^2} + \frac{\mu_5 T^2}{3}\right)\vec{\omega}\,,
\end{aligned} \tag{1.1}$$

where e represents the charge of the fermion, μ is the vector chemical potential, μ_5 is the axial chemical potential, $\vec{B}$ is the magnetic field and $\vec{\omega}$ is the vorticity. These results are only a part of all the anomalous transport coefficients. In particular, we have restricted ourselves to transport associated to the global axial anomaly. However, they are sufficient to gain some intuition about anomalous transport.

The expressions given by (1.1) can be better understood using a qualitative picture in which we analyze whether the fermions of different charges and helicities align or anti-align with the sources. A graphical representation can be found in Figure 1.1, where we treat separately the terms proportional to the magnetic field and the terms proportional to the vorticity. All this discussion fits perfectly the results

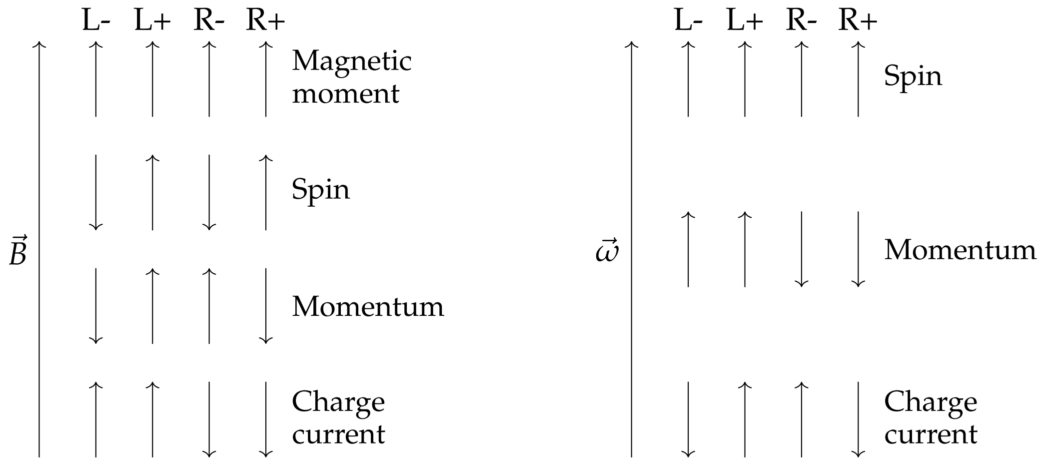

FIGURE 1.1: Qualitative picture of the chiral magnetic and chiral vortical effects through a representation of the alignment or anti-alignment of the fermion vectorial properties with the magnetic field and vorticity for the different charges and helicities.

seen by substituting the charge e by ± 1 and expressing the chemical potentials in terms of the left-handed and right-handed chemical potentials, as $\mu = (\mu_L + \mu_R)/2$ and $\mu_5 = (\mu_L - \mu_R)/2$. The argument and the graphical representation of Figure 1.1 are inspired by Figures 1 and 2 of [139], but we also extend it here to the energy current, noticing that this should be equal to the momentum density.

This qualitative argument works as follows. An external magnetic field tends to align the magnetic moments of all the particles in the fluid. However, the fermions feel this alignment differently depending on their charge and helicity. Those with positive charge have their spin parallel to the magnetic moment and, therefore, parallel to the magnetic field. Those with negative charge, on the contrary, have their spin anti-parallel to the magnetic field. However, helicity makes reference to the alignment of spin and momentum. This means that left-handed fermions possess a motion parallel to their spin while the momenta of right-handed fermions anti-aligns with their spin. The energy current can be simply taken to be the momentum density, because of the symmetry of the energy-momentum tensor i.e. $T_{0i} = T_{i0}$. Thus, positively-charged left-handed and negatively-charged right-handed fermions give a contribution aligned with the magnetic field, while the other two combinations of charge and helicity anti-align. On the other hand, the contribution to the charge current can be found to be proportional to the multiplication of momentum times charge, so in the end the charge current separates the fermions according to their chirality.

If we look at the transport coefficients associated to the vorticity, there is an analogous argument. The spins tend to align with the vorticity. Therefore, the momenta are parallel to the vorticity for left-handed fermions and they are anti-parallel for right-handed fermions. As a result, the energy current, which reduces to the momentum density, behaves like momenta for the different species and separates the fermion according to their chirality. The charge current, however, can be obtained as the charge times the momenta. As a consequence, the current aligns with the vorticity for negatively-charged left-handed fermions and positively-charged right-handed fermions, while it anti-aligns for the other two combinations of charge and helicity.

For a long time, these contributions to the currents were not considered in relativistic hydrodynamics because they required parity breaking. They were first proposed in the late 1970s by Vilenkin [147], who connected them to the parity violation associated to neutrinos in the context of the Standard Model. Fermions in a rotating system were considered in these works and they were found to produce a current along the axis as a result of the coupling of their spin to the angular momentum. Thus, for a chiral imbalance like the one associated to neutrinos this would give rise to a net current. Vilenkin also extended this result to magnetic fields [146] and, after this, there were some sporadic works that studied anomalous transport.

However, not much attention was paid to them until they were rediscovered more than twenty years later in different contexts almost simultaneously. From holography to hydrodynamics, passing by kinetic theory, several works started a new era of interest on the topic. One of the most well-known works is that of Son and Surowka, where a systematic study of the consistency of the hydrodynamic expansion allows them to find that terms proportional to magnetic field and vorticity are not only possible but required in order to satisfy the local version of the second law of thermodynamics [130].

Although it could fix most of the transport coefficients, the entropy current method used in [130] kept some of the coefficients undetermined. In particular, they were the ones proportional to the temperature squared. However, some years later, it was found that the unfixed coefficients were actually related to the mixed gauge-gravitational anomaly [107]. This seemed particularly striking because that part of the anomaly is of higher order in derivatives, so a contribution from it in the low momentum expansion of hydrodynamics was expected to be subleading. With holography, however, the connection to the gravitational contribution of the chiral anomaly was confirmed [108] and it is now a well-established result. A particularly notable aspect of this link to the mixed gauge-gravitational anomaly is that it has been measured in a condensed-matter system recently [57].

In fact, anomaly induced transport appears in many condensed matter systems. For example, a family of systems whose transport phenomena can be explained by the axial anomaly are Weyl semimetals. This exotic state of matter, whose quasiparticles are chiral fermions, presents anomalous Hall effect. Although this transport phenomenon was known before its connection to the anomaly and it is called like that because it does not require the application of an external magnetic field, the naming seems to be premonitory.

The Weyl spinors that describe the quasiparticles of Weyl semimetals always appear in pairs. The band structure of Weyl semimetals, then, is characterized by pairs of cones in which each cone is associated to one of the fermions in a pair. The axial anomaly gives rise to a contribution in the charge current orthogonal to the electric field, and the transport coefficient associated is proportional to the difference between the chemical potentials of the cones in a pair and the separation of the nodes in momentum space. In Chapter 5, we further comment on a holographic model of Weyl semimetals.

However, these are not the only experimental system where anomalous transport is expected to be of crucial importance. The most famous one is actually heavy ion collisions. The interest of heavy ion collisions is that they allow the study of strongly interacting matter in a system that is expected to behave qualitatively different from smaller setups like collisions of protons. The system that results after the collision is typically anisotropic due to the almond shape of the overlapping region of the two colliding objects. However, some experimental results are quite surprising.

First of all, the average shape of the resulting system is indeed that of an almond but, when analyzed per-event, the anisotropy can be extreme. Furthermore, the strongly interacting matter that appears due to the collision of the ions behaves like a fluid and, moreover, it begins to do it a tiny amount of time after the collision happens. Quite surprisingly, it can be very well approximated by ideal hydrodynamics. Thus, it must possess a very small viscosity to entropy ratio, which can be related to a strong coupling among the constituents of the fluid. This ideal fluid of strongly-interacting matter is known as the quark-gluon plasma. It is the part of the QCD phase diagram that appears at high temperature and quark chemical potential [131].

However, it is not clear whether ideal hydrodynamics represents the whole story. The spectators, the parts of the colliding ions that keep their trajectory, produce a strong magnetic field that decreases rapidly in time. In principle, due to the highly out of equilibrium nature of heavy ion collisions, there might be chiral imbalances in the quark-gluon plasma produced by the QCD contribution to the axial anomaly and in that case the chiral magnetic effect would give an important contribution to the transport inside the plasma. So far, no clear experimental evidence seems to confirm the appearance of anomaly induced transport in heavy ion collisions but the results from Relativistic Heavy Ion Collider (RHIC) at Brookhaven seem to be more compatible with it than those from the Large Hadron Collider (LHC) at CERN, in Geneva.

Heavy ion collisions are highly out of equilibrium processes and in Weyl semimetals the out of equilibrium behavior features a very interesting phenomenon called negative magneto-resistivity [129]. Therefore, it might seem reasonable to study nonequilibrium anomalous transport from these applications to experimental setups. However, there is a theoretical motivation too. A result known as Bloch theorem states that any gauge current must vanish in equilibrium [154]. As a result, it seems quite clear that anomalous transport is deeply related to non equilibrium.

Moreover, anomalous transport is also expected to play a significant role in astrophysical systems. Already from the early works about parity violating transport in the 1980s, these currents were connected to the existence of cosmic magnetic fields [139]. Furthermore, anomalous transport has been proposed as an explanation for neutron star kicks [92]. Neutron stars possess much higher velocities than the star from which they form and there are many available explanations for this behavior in the literature. The main issue making it difficult to choose among the different possibilities is that it is still not clear whether there is a correlation between the increase in velocity and any of the other vectorial magnitudes involved, like the spin or magnetic fields. However, the results in [92] could in principle explain the observations even at the quantitative level, giving rise to velocities of the correct order of magnitude.

Let us conclude these enumeration of the applications of anomalous transport to experimentally accessible systems with a very recent one related to quantum information. Since this is very far from the scope of the thesis, we will not comment much on it. However, the basic motivation for proposing anomalous transport as a new tool for quantum computation is exploiting the topological protection and non dissipative nature of the chiral magnetic effect in order to obtain qubits with longer coherence times than the currently available ones [95].

The gauge/gravity duality has proved itself to be a very valuable technique for studying anomalous theories at strong coupling. Anomalies are introduced in holography simply adding Chern-Simons terms to the action [49, 17], and then the usual holographic dictionary can be used to systematically compute n-point functions. As we will see in Section 2.3, retarded Green's functions can be related to linear responses, so in holography weak coupling gravitational computations allow us to compute transport coefficients at strong coupling. Since anomalies do not get renormalized, we do not expect the anomaly induced transport to depend on the energy scale. Furthermore, in this language nonequilibrium is introduced very naturally. It just corresponds to time dependence on the gravity side.

In this thesis we include our work done using the gauge/gravity duality during the four years of PhD. Along the different chapters of the thesis, we make connection between the anomalous transport and the holographic RG flow, we establish the independence of the transport coefficients on momentum relaxation and disorder parameters, we study the holographic model of Weyl semimetals and we analyze the out of equilibrium dynamics of the chiral magnetic effect.

These chapters are simultaneously varied and internally connected, although we could separate them in two parts. The first one is mainly of theoretical interest. It involves Chapters 3 and 4, which are based on [36, 38], respectively. The connection to the holographic RG flow from Chapter 3, which is quite technical due to the construction of conserved currents in gravity through Wald's procedure, allows us to link different arguments from the literature and find a simple way of computing anomalous transport coefficients from quantities evaluated at the horizon. Then, this discussion is supplemented in Chapter 4 with a first nontrivial example where the new contributions to the stress tensor, which were proposed in the previous chapter just from considerations about the niceness of their RG transformations, are shown to give an important contribution. As a result, it can be seen that momentum relaxation and disorder do not have any effect on anomalous transport, which was somehow expected from the nondissipative nature of these phenomena.

The other part has a more applied spirit and it involves Chapters 5 and 6, which are based on [37, 51], respectively. In fact, the former is mainly related to a condensed matter system, Weyl semimetals, while the results of the latter might be useful for the phenomenology of the quark-gluon plasma. As already discussed above, Weyl semimetals present anomalous Hall effect. We can thus propose a holographic model that describes the physics of Weyl semimetals at strong coupling. This was done by our research group some months before this PhD started and in Chapter 5 we include some further work on the model. In particular, the universality of the bottom-up model is checked and the results about the anomalous Hall effect are extended to its axial counterpart. In Chapter 6, on the other hand, we introduce out of equilibrium dynamics in holography through Vaidya metrics with momentum relaxation and compute the one-point functions of the charge currents. The main result is that the currents take some time before they build up. Since at higher energies the magnetic fields in the quark-gluon plasma decay earlier, we propose that such a lapse of time before there is an anomaly induced contribution to the current might give an explanation of why the experimental results from RHIC at Brookhaven seem to be more compatible with the appearance of anomalous transport than those from the LHC at CERN, in Geneva.

The projects were not developed in the order they are presented here, but we have decided to place them in such a way that the logic of the thesis is the best possible. The publication on which Chapter 3 is based appeared later than the one associated to Chapter 4, but the developments of membrane currents from Chapter 3 were crucial to conclude the fact that anomalous transport coefficients are independent of the introduction of momentum relaxation. Chapter 5 is based on my first publication, which appeared at a time when the model of holographic Weyl semimetals had just been proposed. However, it has a more applied nature so it connected better with the last project. This last project, finally, gives rise to Chapter 6, where we include the extension to out of equilibrium contexts.

Let us close this introduction with some comments about the general structure of the thesis. Besides the chapters about the projects, the thesis has an introduction to the theoretical background necessary to understand anomalous transport in holography in Chapter 2 and some concluding remarks in Chapter 7. The chapters about our original work are based on one publication each, as discussed above, and they all possess the same structure. First, they include a section called Motivation, where we explain the reasons to begin the project and we relate it to other approaches in the literature. After that, we introduce the holographic model, the techniques used and all the details about the computations. Then, we report the results. Finally, all those chapters end with a section called Discussion, where we comment on the results and make some per-project concluding remarks.

Chapter 2

Theoretical foundations

All the projects covered in this thesis revolve around the same physical phenomena: anomalous transport [102]. In order to understand the context of this work, then, one needs to understand anomalies, relativistic hydrodynamics and linear response theory. In the current chapter we will cover those topics and also include a brief introduction to the formal aspects of the gauge/gravity duality.

2.1 Anomalies in quantum field theory

In modern day theoretical physics there is a big emphasis on the understanding of symmetries. They are related through Noether's theorem to conserved quantities, but they also allow to classify different models and establish dualities between them. In some cases quantum mechanical corrections can break those symmetries and in those situations we refer to this breaking as an anomaly. Characterizing anomalies properly is fundamental to understand the physics of the theory. Let us imagine, for example, a theory that is scale invariant. If there is a breaking of the symmetry at the quantum level, then the coupling constants will be allowed to run along the renormalization group flow, giving rise to different dynamics at different energy scales. This is for example what produces in QCD such different phenomena as asymptotic freedom and confinement.

Although they are usually thought to be, anomalies are not necessarily pathological. The former example exemplifies this perfectly, where the physics of different scales only arise due to an anomaly. We could say, in general, that they are not problematic for global symmetries. In those cases, the anomaly just means that the symmetry is no longer present at the quantum level. No particularly profound property of the theory is broken.

On the contrary, anomalies associated to gauge symmetries are disastrous and they have to be canceled at all costs. We will meet this type of anomalies at several points in this section and we will consistently insist on the necessity to cancel them. The reason why they are a big problem is that gauge symmetry is not a regular symmetry. It is a redundancy included in order to keep our theories manifestly local and Lorentz invariant. However, the price to pay is the introduction of negative or zero norm states. The contribution of these degrees of freedom has to be canceled in order to keep unitarity, but if the anomaly breaks gauge symmetry this is no longer possible, making the theory inconsistent. The solution then reduces to canceling the anomaly, which imposes nontrivial constraints on the theory.

We will only be concerned here about the axial and chiral anomalies, since they are the ones that have associated transport phenomena. The axial anomaly, which was the one first encountered by Adler, Bell and Jackiw, is a global anomaly, while the chiral anomaly is a gauge anomaly. We will treat them separately in order to make this distinction clear. After that we will briefly comment on how they change for nonabelian theories and how one can use global symmetries to constraint the spectrum of the theory. Then we briefly discuss the gravitational anomalies and, finally, we introduce the consistency conditions and the descent equations, that allow to obtain the anomalies in a systematic way from a differential geometry reasoning. At the end of this section, we discuss the different definitions of the currents .

There are more topics that could be covered, like Fujikawa's method or the relation to index theorems. However, we have restricted ourselves to the more basic topics necessary to understand the interest of our work. For more complete reviews, the reader might resort to the books and reviews available in the literature [25, 26, 53, 72]

2.1.1 The global axial anomaly

The axial anomaly was the one that debuted the study of anomalies in quantum field theory. It first appeared in the literature as a contradiction between the perturbative computation of the decay rate of a neutral pion into two photons and the expected vanishing of the effective coupling constant of that decay from combining gauge symmetry and the partially conserved axial current hypothesis [24, 1]. We review here their result, as a prologue to discussing the more interesting chiral gauge anomalies in the Section 2.1.2.

Let us consider quantum electrodynamics (QED) in 4 dimensions

$$S_{QED} = \int d^4x \left[\bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e\bar{\psi}\gamma^\mu A_\mu \psi \right] . \tag{2.1}$$

It presents the well-known $U(1)$ **gauge symmetry**

$$\psi \to e^{i\alpha}\psi \,, \quad A_\mu \to A_\mu + \partial_\mu \alpha \,,$$

and it thus has an associated conserved current

$$J^\mu = \bar{\psi}\gamma^\mu\psi \,,$$

the electromagnetic current. This current satisfies the continuity equation

$$\partial_\mu J^\mu = 0 \,, \tag{2.2}$$

which assures charge conservation. Massless QED also has a **global axial symmetry**

$$\psi \to e^{i\alpha\gamma_5}\psi \,,$$

whose associated current is

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \,.$$

This current satisfies in general the following continuity equation

$$\partial_\mu J_5^\mu = 2im\bar{\psi}\gamma_5\psi\,. \tag{2.3}$$

This identity, known as the **pseudovector-pseudoscalar equivalence**, shows explicitly that it is only a classically conserved current in the massless limit. Both currents are quantities that at the quantum level are to be conceived as composite operators. Anomalies in this context will appear if either (2.2) or (2.3) get a quantum correction. The conservation of the axial current is not crucial for the survival of the theory. However, the vector current must remain conserved after renormalization because it is, as discussed above, associated to a true gauge symmetry.

Let us concentrate on the axial current. We take the electromagnetic field in (2.1) to be a external field. As a result, the kinetic Maxwell term vanishes and the interaction term in the action will be the only surviving one involving A_μ. Performing a perturbative expansion in small coupling constant e in the path integral formalism, we can find an expression for the quantum corrections to the axial current's conservation

$$\begin{aligned}\langle \partial_\mu J_5^\mu \rangle =& \frac{1}{Z}\int D\psi D\bar{\psi}\, \partial_\mu J_5^\mu \, e^{i\int d^4x\left[\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - eJ^\mu A_\mu\right]} \\ =& \frac{1}{Z}\int D\psi D\bar{\psi}\, \partial_\mu J_5^\mu \, e^{iS_f}\left[1 - ieJ^\nu A_\nu - \frac{e^2}{2}J^\nu A_\nu J^\rho A_\rho + \mathcal{O}(e^3)\right]\,,\end{aligned}$$

where Z stands for the full QED partition function, S_f stands for the massive fermion action and we have rewritten the last term in (2.1) as $-eJ^\mu A_\mu$. We are deliberately abusing notation in this expression in order to avoid cluttering. Please note, however, that every $J^\mu A_\mu$ depends on a different spacetime coordinate and there is an associated integral over such coordinate with each of these terms, as will be clear below in (2.4).

It can be seen by use of properties of the gamma matrices that only traces involving an odd number of gamma matrices will contribute in the presence of the γ^5 from J_5^μ, so the first correction is given by the term with e^2 in the previous expansion

$$\langle \partial_\mu J_5^\mu(x_1)\rangle = -\frac{e^2}{2}\int d^4x_2 \int d^4x_3 \langle \mathrm{T}\left[\partial_\mu J_5^\mu(x_1) J^\nu(x_2) J^\rho(x_3)\right]\rangle A_\nu(x_2) A_\rho(x_3) + \mathcal{O}(e^3)\,, \tag{2.4}$$

where T stands for time ordering. Using Feynman diagrams, it can be seen that this contribution is given by the well-known **triangle diagrams** in Fig. 2.1. Since A is now an external field, the interpretation of the path integral as the triangle diagrams amounts to using the Wick theorem to contract the spinors from the definitions of the currents and keeping two external field insertions. The computation of the triangles is already covered in many reviews and books, so we will only comment on the interesting aspects of it.

The integrals that have to be performed are ambiguous because they are linearly divergent. They depend on the labeling of the fermion loop momenta and there is in principle total freedom to choose the labeling. However, there is a simple procedure exploiting symmetries that allows to fix the loop momenta labeling from conditions on the external legs and get the final result.

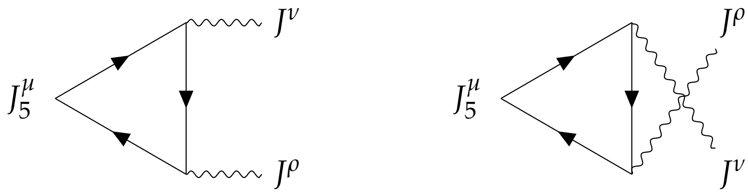

FIGURE 2.1: Triangle diagrams contributing to the axial anomaly for an abelian gauge field in four dimensions. The diagrams were created using *TikZ-Feynman* [48].

The first step is using Lorentz invariance to write down all the possible tensor structures that contain the two independent external momenta and one Levi-Civita tensor. The final result of the computation in momentum space will necessarily have this structure and we know there will be a Levi-Civita symbol present because of having a γ_5 in the amplitudes. Therefore, we have turned the computation of the triangles into the computation of the coefficients accompanying the different tensor structures. Naively, the number of undetermined coefficients is eight, but applying the fact that the amplitudes should be invariant under the exchange of the two vector currents reduce them to four. Finally, from dimensional analysis it becomes obvious that only one of the four coefficients involves a divergent integral.

At this point we need to impose some further symmetry in order to obtain a well-defined result. Certainly, it seems reasonable to impose gauge invariance with respect to the electromagnetic $U(1)$ symmetry, because we have discussed we want to preserve this symmetry. The Ward identities tell us that the two external momenta associated to the electromagnetic currents must be transverse to the amplitude, and this fixes the labeling of the internal momenta. In fact, this condition constrains the value of the divergent coefficient in terms of the other coefficients and all the independent integrals become convergent in the end. One gets to the final result, which in real space reads

$$\langle \partial_\mu J_5^\mu \rangle = 2im\langle \bar{\psi}\gamma_5\psi \rangle + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \,. \tag{2.5}$$

This is precisely the famous result known as **Adler-Bell-Jackiw anomaly** [24, 1].

Let us go back to the freedom of choice for the loop momenta. If one performs the computation in general, with arbitrary shifts of the internal momenta proportional to the external ones, it is straight-forward to check that one cannot preserve at the same time the electromagnetic gauge symmetry and the global axial symmetry. This will be much better understood in the next section when we discuss gauge anomalies.

The anomaly (2.5) appears at first order in $\hbar$ because it is a one-loop computation. However, one could wonder if there are also nonvanishing contributions to the axial current's one-point function from diagrams with higher number of loops. It was shown by Adler and Bardeen that those diagrams do not have any contribution in their celebrated **Adler-Bardeen nonrenormalization theorem** [2].

2.1.2 The chiral anomaly

In this work we are interested on transport induced by the chiral anomaly, so we will now move on to the study of this anomaly. It will also help us better understand the anomaly associated to the global axial current. Hopefully, all the obscure aspects of the previous section will become clear now.

The notion of chirality applies in general to phenomena that are not equivalent to their mirror image. In the context of quantum field theory, it appears as a well-defined notion for fermions in even dimensions, since the Lorentz group possesses two unitarily inequivalent spinor representations in those situations. This produces two different types of spinors. They receive the name of left- and right-handed because in the massless limit they present definite helicity, which is the projection of spin onto momentum. Parity transformations exchange one representation by the other, reversing helicity. Therefore, if the theory is parity-invariant, there will be the same number of left- and right-handed fermions, so chiral anomalies will only arise in theories that break parity.

In the massless limit the theory is symmetric under transformations in which the two sectors transform with different phases. The chiral anomaly shows up precisely because at the quantum level only a linear combination of those two rotations can be a symmetry. Then, we will begin our discussion considering massless fermions in the chiral representation of the gamma matrices in 4 dimensions

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix},$$

where $i = 1,2,3$ and σ_i stands for the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The definition of the γ^5 allows us to define the projectors

$$\begin{aligned} \psi_L &= P_L\psi = \frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right)\psi = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}\psi, \\ \psi_R &= P_R\psi = \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right)\psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}\psi. \end{aligned}$$

We now take a theory of massless fermions which are coupled chirally to two external gauge fields. One of the gauge fields, L_μ, couples only to the left-handed fermion with charge q_L and the other gauge field, R_μ, couples only to the right-handed fermion with charge q_R. The choice of this action stems from an educated guess that will allow us to study the two chiral symmetries in play. The action reads

$$S = \int d^4x \left[i\bar{\psi}\gamma^\mu\partial_\mu\psi - q_L\bar{\psi}\gamma^\mu L_\mu P_L\psi - q_R\bar{\psi}\gamma^\mu R_\mu P_R\psi \right].$$

As already advanced in the text above, this action possesses two classical gauge symmetries. The first one is given by transformations

$$\psi \to P_R\psi + e^{iq_L\alpha_L}P_L\psi\,, \quad L_\mu \to L_\mu + \partial_\mu\alpha_L\,,$$

and its associated conserved current is

$$J_L^\mu = q_L\bar{\psi}\gamma^\mu P_L\psi\,.$$

The other symmetry is defined as

$$\psi \to P_L\psi + e^{iq_R\alpha_R}P_R\psi\,, \quad R_\mu \to R_\mu + \partial_\mu\alpha_R\,,$$

and its associated conserved current is

$$J_R^\mu = q_R\bar{\psi}\gamma^\mu P_R\psi\,.$$

Both of them satisfy the following classical conservation equations

$$\partial_\mu J_L^\mu = \partial_\mu J_R^\mu = 0\,.$$

Now, in order to analyze the anomaly, we need to check how these last equations change at the quantum level. For simplicity, let us treat both currents separately.

We first consider the case in which q_R vanishes. There will only be one gauge symmetry present and, thus, only one current. From the perturbative expansion at small q_L of the path integral expression of $\langle\partial_\mu J_L^\mu\rangle$, we arrive at an equation analogous to (2.4) where, instead of an axial current and two vector currents in the time-ordered correlator, we have three left-handed currents. The chiral anomaly for the left-handed current is thus given also by a triangle. Now, the three vertices are associated to the left-handed coupling and the amplitude then comes with a prefactor q_L^3. If we had considered vanishing q_L and nonvanishing q_R, everything would look the same, except for the gauge couplings and projectors involved in the definition of the right-handed currents. Those changes only amount to a slightly different prefactor $-q_R^3$.

The computation of the rest of the amplitude is essentially the same for both cases, and analogous to the one for the axial anomaly in previous section. Projection operators have the property that applying them twice yields the same result as applying them once, $P_{L,R}^2 = P_{L,R}$, so only one out of the three projectors survives in each of the amplitudes. Besides that, we already commented on the fact that the presence of one γ^5 is necessary for the amplitude not to vanish, so we can drop the unity matrices in the surviving projection operators. The origin of the different sign in the prefactors of the amplitudes is thus the sign of the γ^5 in the projectors.

The ambiguities present in the computation of the global anomaly persist. However, the criterion to fix the labeling of the loop momenta is not obvious here, because we do not have the same symmetries. In the previous case, it was rather obvious that the electromagnetic symmetry was the one that had to remain present, while the axial symmetry could be broken at the quantum level. Here, we are treating the two different symmetries one at a time and both are true gauge symmetries.

Therefore, there are two possibilities that could seem physically plausible. We can impose Bose symmetry among the three legs because the inserted currents are the same for the three vertices. This gives the result

$$\langle \partial_\mu \mathcal{J}^\mu_{L,R} \rangle = \pm \frac{q^3_{L,R}}{96\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \,,$$

which is known as the **consistent anomaly**. Please note that we include a generic $F_{\mu\nu}$ to avoid cluttering, but this field-strength is the curvature associated to L_μ or R_μ depending on each case.

We could also impose this current to couple covariantly to the external gauge fields. In that case we need to enforce the Ward identity (the amplitude is transverse to the external momenta) in two of the three vertices. This gives the result

$$\langle \partial_\mu J_{L,R} \rangle = \pm \frac{q^3_{L,R}}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \,, \tag{2.6}$$

and it is known as the **covariant anomaly**. More on the distinction between the consistent and the covariant form of the anomaly will be said in Section 2.1.7. However, at this point we can already notice that anomalies allow for different definitions of the quantum currents.

Let us now revisit the axial anomaly. If we consider the case in which the chiral gauge couplings are equal for left- and right-handed fermions ($q_L = q_R \equiv q$), the action can be rearranged as

$$S = \int d^4x \left[i\bar{\psi}\gamma^\mu \partial_\mu \psi - q\bar{\psi}\gamma^\mu \left(\frac{R_\mu + L_\mu}{2} \right) \psi - q\bar{\psi}\gamma^\mu \left(\frac{R_\mu - L_\mu}{2} \right) \gamma^5 \psi \right] .$$

We interpret the interaction terms as couplings to vector and axial gauge fields

$$V_\mu = \frac{R_\mu + L_\mu}{2} \,, \qquad A_\mu = \frac{R_\mu - L_\mu}{2} \,. \tag{2.7}$$

Furthermore, we can define vector and axial currents as the quantities that couple in this action to the external vector and axial fields, thus giving a definition of them formally equal to the one obtained from QED in the previous section. Please note that the gauge field that is equivalent to previous section's A_μ is V_μ. The same notation with a vector and an axial field we are using here will be again used in Chapters 5 and 6.

The vector and axial currents have an expression in terms of the left-handed and right-handed consistent currents which reads

$$\mathcal{J}^\mu = \mathcal{J}^\mu_R + \mathcal{J}^\mu_L \,, \qquad \mathcal{J}^\mu_A = \mathcal{J}^\mu_R - \mathcal{J}^\mu_L \,. \tag{2.8}$$

As will be later explained, they are expressed in terms of consistent currents because they come from the variation of the action. Using the previous expressions of the consistent anomaly, we can write the anomaly in this basis for a single fermion with charge $q = 1$ as

$$\begin{aligned}\langle \partial_\mu \mathcal{J}^\mu \rangle &= -\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F^5_{\rho\sigma}\,,\\ \langle \partial_\mu \mathcal{J}^\mu_5 \rangle &= -\frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(F_{\mu\nu}F_{\rho\sigma} + F^5_{\mu\nu}F_{\rho\sigma}\right)\,,\end{aligned}$$

where $F_{\mu\nu}$ is the field-strength of V_μ and $F^5_{\mu\nu}$ is the field-strength of A_μ. This result looks worrisome because we have two gauge symmetries that are broken at the quantum level. However, while before we had two gauge symmetries that were sort of equivalent, now we have an asymmetry between them.

Fortunately, each symmetry has a clear physical meaning and we can exploit that to interpret this result. It essentially boils down to the fact that we have to cancel the anomaly on the vector gauge symmetry at all cost. This is the current that couples to Maxwell's equations and consistency of the equations imposes vanishing of the divergence of the current. However, we have in principle no need to cancel the anomaly for the axial one so it is safe to drop axial symmetry as a true gauge symmetry. Once we do this, we can redefine our action, and currents, with terms that would otherwise be gauge dependent.

These terms that we add to the effective action are called **Bardeen counterterms** and in general they read

$$\Gamma_B = \int d^4x \epsilon^{\mu\nu\rho\sigma}V_\mu A_\nu \left(aF_{\rho\sigma} + bF^5_{\rho\sigma}\right)\,.$$

The variation with respect to the vector field

$$\int d^4x \epsilon^{\mu\nu\rho\sigma}\delta V_\mu \left(A_\nu \left(2aF_{\rho\sigma} + bF^5_{\rho\sigma}\right) - aV_\nu F^5_{\rho\sigma}\right)\,,$$

gives a term that is now added to $\mathcal{J}^\mu$. Taking the divergence of this new contribution to the current, it is quite straight-forward to check that we must take $a = -\frac{1}{12\pi^2}$ and $b = 0$ in order to cancel the anomaly of the vector current.

We can now take the variation of the counterterm with respect to the axial field in order to see how the axial anomaly changes

$$\int d^4x \epsilon^{\mu\nu\rho\sigma}\delta A_\mu \left(-V_\nu \left(aF_{\rho\sigma} + 2bF^5_{\rho\sigma}\right) + bA_\nu F_{\rho\sigma}\right)\,.$$

Taking the divergence of this last term and substituting the values of a and b obtained above, we can get to the final expression of the **axial anomaly**

$$\begin{aligned}\partial_\mu\langle \mathcal{J}^\mu \rangle &= 0\,,\\ \partial_\mu\langle \mathcal{J}^\mu_5 \rangle &= -\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(F_{\mu\nu}F_{\rho\sigma} + \frac{1}{3}F^5_{\mu\nu}F^5_{\rho\sigma}\right)\,.\end{aligned}$$

Please note that the first term of this form of the axial anomaly is equal to the one we obtained for the global axial symmetry.

2.1.3 Extension to the non-abelian case

In this section, we cover how the results change for the non abelian case. We first consider the counterpart of the global axial anomaly to then move on to the chiral anomaly. Only the second one is pathological so we will also discuss the anomaly cancellation conditions at the end of the section.

We consider a fermion theory with an external non abelian gauge field. It is very similar to the QED action (2.1), except for the fact that now the gauge field comes with a generator of the algebra which is also present in the definition of the conserved current, according to

$$A_\mu = A^a_\mu T^a\,, \qquad J^\mu = \bar{\psi}\gamma^\mu T^a \psi\,.$$

Similarly to QED, global axial transformations are also a classical symmetry in the massless limit. They possess the same associated conserved current and the latter also satisfies the pseudovector-pseudoscalar equivalence (2.3).

When one tries to compute the axial anomaly, the first diagrams involved are the same ones as for the abelian case, up to group factors that appear due to the non-abelian coupling in the vector vertices. The computation of these triangle diagrams works in the same way as the abelian case, too. However, the result is gauge dependent. It becomes then obvious that one needs to also include the next diagram: a box diagram with three vector vertices. The final gauge invariant result reads

$$\langle \partial_\mu J^\mu_5 \rangle = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu \mathrm{Tr}\left(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\right)\,,$$

and is known as the **singlet anomaly**. The three diagrams involved are the ones in Fig. 2.2 and there is no further contributions to the global anomaly.

This form of the anomaly, written as the exterior derivative of the Chern-Simons 3-form, can be arranged to be formally equal to the abelian anomaly but with the non abelian field strengths. Moreover, the diagrams have a property with deep consequences. There is one more diagram than for the abelian case but it comes with a prefactor that is equal to the one for the triangles. Therefore, anomaly cancellation imposes only one constraint for the nonabelian case.

Let's now move on to the gauge anomaly for the nonabelian case. For simplicity, we consider the action of a Dirac fermion coupled to external non abelian left and right fields with the same charge $q = 1$. The action reads

$$S = \int d^4x \left[i\bar{\psi}\gamma^\mu\left(\partial_\mu - iL^a_\mu T^a\right)P_L\psi + i\bar{\psi}\gamma^\mu\left(\partial_\mu - iR^a_\mu T^a\right)P_R\psi\right]\,.$$

In this theory there are two different gauge fields with associated symmetry transformations

$$\begin{aligned}
\psi &\to e^{i\alpha^a_L T^a}P_L\psi + e^{i\alpha^a_R T^a}P_R\psi\,,\\
L_\mu \equiv L^a_\mu T^a &\to e^{i\alpha^a_L T^a}\partial_\mu e^{-i\alpha^a_L T^a} + e^{i\alpha^a_L T^a}L_\mu e^{-i\alpha^a_L T^a}\,,\\
R_\mu \equiv R^a_\mu T^a &\to e^{i\alpha^a_R T^a}\partial_\mu e^{-i\alpha^a_R T^a} + e^{i\alpha^a_R T^a}R_\mu e^{-i\alpha^a_R T^a}\,.
\end{aligned}$$

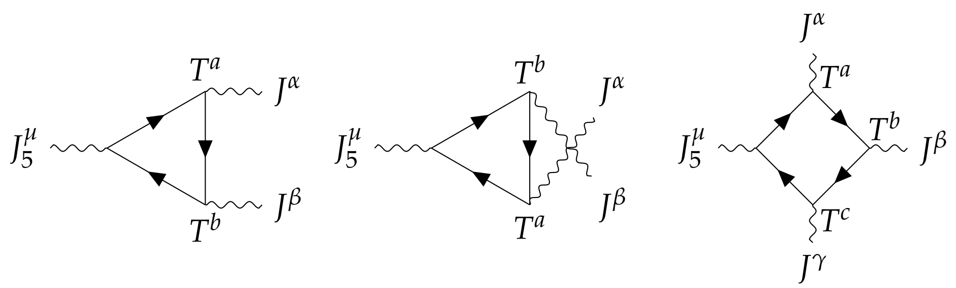

FIGURE 2.2: Triangle and box diagrams contributing to the global axial anomaly for a nonabelian gauge field in four dimensions. The symbols at the vertices of the loops stand for the gauge factors that need to be included for the vector couplings. The diagrams were created using *TikZ-Feynman* [48].

However, the main new ingredient is that the conservation of the current is no longer the same, since the currents are only conserved with respect to the covariant derivative. Therefore, the equations whose quantum corrections we need to analyze in order to study the nonabelian chiral anomaly read

$$\partial_\mu (J_L^a)^\mu + f^{abc} L_\mu^b (J_L^c)^\mu = 0\,,$$
$$\partial_\mu (J_R^a)^\mu + f^{abc} R_\mu^b (J_R^c)^\mu = 0\,,$$

where f^{abc} is defined by the algebra $\left[T^a, T^b\right] = i f^{abc} T^c$.

It might be convenient to use the basis introduced in (2.7) and (2.8) with the appropriate extension to the non abelian case. In terms of these new fields and currents, the conservation equations read

$$D_\mu (J^a)^\mu \equiv \partial_\mu (J^a)^\mu + f^{abc} V_\mu^b (J^c)^\mu + f^{abc} A_\mu^b (J_5^c)^\mu = 0\,,$$
$$D_\mu (J_5^a)^\mu \equiv \partial_\mu (J_5^a)^\mu + f^{abc} V_\mu^b (J_5^c)^\mu + f^{abc} A_\mu^b (J^c)^\mu = 0\,.$$

We start the computation of the quantum corrections to this last equation by expanding perturbatively in a way analogous to (2.4), which tells us which diagrams have to be calculated. In the abelian case, as we saw in (2.4), only terms with two gauge fields contributed to the triangle because the third vertex was associated to the divergence. Here, the triangle will also get contributions with three gauge fields. Furthermore, there will be nonvanishing contributions from the box diagram and also from the pentagon. The triangle is still linearly divergent, but the box is logarithmically divergent and the pentagon is convergent. All the new contributing diagrams are depicted in Fig. 2.3, where exchange of the different equivalent legs has to be performed in order to obtain the consistent anomaly.

These diagrams involve group theory factors coming from the different vertices. Interestingly, as it happens for the global case, the group theory factor of the rest of the diagrams is proportional to the one coming from the triangle. Therefore, the anomaly could be canceled through the vanishing of the triangle prefactor. For

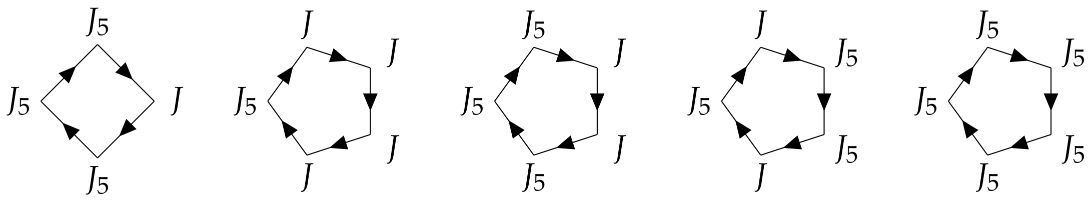

FIGURE 2.3: Diagrams that have to be added to the ones in Fig. 2.2 in order to compute the Bardeen anomaly in four dimensions. This is a schematic representation. The appropriate group factors and external momenta indexes should be included for the different vertices. The diagrams were created using *TikZ-Feynman* [48].

nonabelian groups the anomaly coefficient reads

$$\mathrm{Tr}\left[T^a\left\{T^b, T^c\right\}\right] = 0\,.$$

In fact, in those cases in which there are many chiral fermions, this anomaly cancellation condition could constraint the possible charges of the particles in the spectrum of the theory.

In this vector-axial basis we must again impose that the vector current is conserved. A Bardeen counterterm, with a more complicated expression than the one for the abelian case, can be added to the effective action in order to achieve that goal. Once this ambiguity is fixed, the expression for the nonabelian axial anomaly is

$$\begin{aligned}
\langle D_\mu(\mathcal{J}^a)^\mu\rangle =&0\,,\\
\langle D_\mu(\mathcal{J}_5^a)^\mu\rangle =&-\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\Bigg[T^a\Bigg(F_{\mu\nu}F_{\rho\sigma}+\frac{1}{3}F^5_{\mu\nu}F^5_{\mu\nu}\\
&+\frac{8i}{3}\left(A_\mu A_\nu F_{\rho\sigma}+A_\mu F_{\nu\rho}A_\sigma+F_{\mu\nu}A_\rho A_\sigma\right)-\frac{32}{3}A_\mu A_\nu A_\rho A_\sigma\Bigg)\Bigg]\,,
\end{aligned}$$

which is popularly known as the **Bardeen anomaly** [18]. Please note that the first two terms are formally equal to the abelian case, with the new field strengths

$$\begin{aligned}
F_{\mu\nu} =&\partial_\mu V_\nu-\partial_\nu V_\mu-i\left[V_\mu, V_\nu\right]-i\left[A_\mu, A_\nu\right]\,,\\
F^5_{\mu\nu} =&\partial_\mu A_\nu-\partial_\nu A_\mu-i\left[V_\mu, A_\nu\right]-i\left[A_\mu, V_\nu\right]\,.
\end{aligned}$$

We have expressed all our results in terms of the consistent currents. Analogously to the abelian case, however, the properties of the anomaly are obscure in this vector-axial basis. We will not include here the expressions for the left- and right-handed currents because they do not add much insight, but in terms of those currents the consistent anomaly is obtained imposing Bose symmetry. As a result, it would become obvious that the consistent anomaly is not covariant, as it cannot be expressed only in terms of the field strengths of the two gauge fields. After choosing the appropriate Bardeen counterterm, though, the result in this vector-axial basis preserves vector gauge symmetry i.e. it depends on $F_{\mu\nu}$ but not V_μ.

2.1.4 Anomaly matching condition

There is a result about anomalies with very deep phenomenological implications, the so-called **anomaly matching condition** [77]. In particular, it uses global symmetries to pose nontrivial constraints on theories that have different degrees of freedom in the UV and the IR, like confining theories such as QCD or technicolor. It gives nonperturbative information of the theory, like constraints on the matter spectrum at different energy scales or signals of the appearance of spontaneous symmetry breaking at low energies. Although it won't be used in the rest of this thesis, let us point out to the most important aspects of the proof by 't Hooft, so that the reader can fully grasp the implications of this result.

We consider a theory with a gauge (local) symmetry group G_l and a global symmetry group G_g. In the context in which the matching condition was proposed, the gauge theory was QCD and the global symmetry was flavor. The G_l^3 anomaly has to be canceled for the gauge theory to respect unitarity and the mixed anomaly $G_g \times G_l^2$ should also be zero if we want to keep the global symmetry. However, in principle there is no constraint on the value of the G_g^3 anomaly. If we weakly gauge the global symmetry by coupling its Noether current to a gauge field, though, we will also be forced to cancel it. In that case, we would be forced to supplement the spectrum of the theory with some spectator fermions which would only be charged under G_g. Nonetheless, we can guarantee the fields do not alter the dynamics of the original theory by keeping the gauge coupling small.

In the IR, the relevant degrees of freedom are the G_l bound states and the spectator fields. However, the latter give the same contribution in the UV and in the IR because they are weakly coupled. In fact, the matching condition still holds in the limit when the gauge coupling goes to zero and the spectator fields decouple. Therefore, the global anomaly of the microscopic degrees of freedom have to match those of the IR composite fermions. If the global symmetry were spontaneously broken at low energies, then the same arguments would still hold but the Goldstone bosons would also have a contribution.

In fact, from this work a new terminology for anomalies has been introduced in the literature. The name **'t Hooft anomalies** refers to those anomalies which are associated to global symmetries that, precisely because they are anomalous, cannot be gauged.

2.1.5 Gravitational anomalies

Gravity is another gauge theory. If we expand around a flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} ,$$

diffeomorphisms are generated by a vector gauge parameter

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{2} \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu \right) .$$

It is straight-forward to check that invariance under these gauge transformations amounts to the conservation of the energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0\,.$$

Thus, a gravitational anomaly would arise if this conservation equation obtains quantum corrections. Since this is a gauge symmetry, its associated anomaly has to be canceled.

In principle, gravity does not couple differently to fermions with different helicity. However, CPT transformations must be a symmetry of any physically viable theory. These transformations reverse the helicity of fermions for $4k$ dimensions and they preserve it for $(4k+2)$ dimensions. Thus, analogously to the discussion that chiral anomalies only arise in parity breaking theories, gravity will have a contribution to the chiral anomaly only in $4k$ dimensions. In $(4k+2)$ dimensions, though, pure gravitational anomalies can appear. We will only consider the former case here.

The **gravitational contribution to the chiral anomaly** is actually a mixed anomaly. We can therefore use a Bardeen counterterm to impose the cancellation of the diffeomorphism anomaly at the expense of a nonvanishing anomaly for the chiral currents, much in the same way we did it for the vector and axial anomalies. The role of the gauge field is played here by the Levi-Civita connection and the role of the field strength is played here by the Riemann tensor.

Then, when one performs the perturbative expansion in the path integral expression for the one-point function of the divergence of the chiral current in a theory of a fermion coupled to an external graviton, it can be seen that the first contribution is given by a triangle where the other two vertices are populated by energy-momentum tensors. The result for the gravitational contribution to the chiral anomaly is

$$\langle \nabla_\mu \mathcal{J}^\mu_{L,R} \rangle = \pm \frac{1}{768\pi^2} \epsilon^{\mu\nu\rho\sigma} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\sigma}\,. \tag{2.9}$$

The triangle will only include one generator of the non abelian gauge theory, so the cancellation condition in a theory with N_R right-handed fermions and N_L left-handed fermions is

$$\sum_{i=1}^{N_R} \mathrm{Tr} T^a_R - \sum_{i=1}^{N_L} \mathrm{Tr} T^a_L = 0\,.$$

If the gauge field is abelian, this reduces to the sum of the right-handed charges minus the sum of the left-handed charges. Interestingly, this anomaly cancellation condition gives rise to nontrivial constraints for the viable models of nature, like the one on the hypercharges of the Standard Model [4].

2.1.6 Consistency conditions and descent equations

Let us begin with the fermion effective action

$$e^{i\Gamma[A]} = \int D\psi D\bar{\psi} \exp\left[i \int d^d x \bar{\psi} \gamma^\mu \left(i\partial_\mu - A_\mu \right) \psi \right] \,,$$

where the gauge field is $A_\mu = A_\mu^a T^a$. Considering that functional derivation of the effective action with respect to the gauge field inserts minus the current operator $J^{\mu,a} = \bar{\psi}\gamma^\mu T^a \psi$, if we now take the gauge variation of the effective action the result is

$$\delta_\epsilon \Gamma = \int d^d x \frac{\delta \Gamma}{\delta A_\mu^a} \delta_\epsilon A_\mu^a = -\int d^d x \langle J^{\mu,a} \rangle \delta_\epsilon A_\mu^a .$$

The gauge variation of the gauge field reads

$$\delta_\epsilon A_\mu^a = \partial_\mu \epsilon^a + f^{abc} A_\mu^b \epsilon^c \equiv (D_\mu \epsilon)^a ,$$

so substituting it in the previous equation, integrating by parts and recognizing the anomaly as the divergence of the current's one-point function, we can finally obtain the relation between the effective action and the anomaly

$$\delta_\epsilon \Gamma = \int d^d x \epsilon^a D_\mu \langle J^{\mu,a} \rangle \equiv \int d^d x \epsilon^a \mathcal{A}_a[A] .$$

The anomaly is then the gauge variation of the fermion effective action. Thus, one should in principle be able to compute chiral anomalies by constructing the appropriate effective actions.

The composition of two infinitesimal gauge transformations has to satisfy that

$$\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1} = \delta_{[\epsilon_1,\epsilon_2]} .$$

Applying this relation to the effective action sets constraints on the possible anomalies

$$\int d^d x \epsilon_2^a \delta_{\epsilon_2} \mathcal{A}_a - \int d^d x \epsilon_1^a \delta_{\epsilon_1} \mathcal{A}_a = \int d^d x \, [\epsilon_1, \epsilon_2]^a \, \mathcal{A}_a ,$$

which are called **Wess-Zumino consistency conditions** [150].

If a generic effective action Γ is a local functional, then there will be no anomaly associated to it, because it could be canceled by the addition of local counterterms. Any potential anomaly that is the variation of a functional automatically satisfies the consistency conditions. However, only if the functional is nonlocal we obtain nontrivial solutions to the consistency conditions and we can really talk about anomalies.

It can be seen that a nontrivial solution to the consistency conditions is generated by invariant polynomials in an even-dimensional ($D = 2m$) space

$$P(F) = \sum_{np \leq m} c_{n,p} (\mathrm{Tr} F^n)^p ,$$

where F stands for the field strength of the gauge field. These quantities possess nice mathematical properties: (1) they are closed forms, and (2) they are gauge invariant. The former means, thanks to the Poincaré lemma, that they are also locally exact

$$\mathrm{Tr} F^n = d\omega_{2n-1} ,$$

and ω_{2n-1} is the Chern-Simons (2n-1)-form. With the help of the second property and this last equation, it can also be shown that the gauge variation of the Chern-Simons form is closed and, therefore, locally exact

$$\delta_\epsilon \omega_{2n-1} = d\alpha_{2n-2} \,. \tag{2.10}$$

Going back to the Wess-Zumino consistency conditions, we can take the required (2n-2)-dimensional nonlocal action to be given, up to a constant c_n which can eventually be found matching the perturbative computation, by the integral of the Chern-Simons (2n-1)-form

$$\Gamma[A] = c_n \int_{\mathcal{M}_{2n-1}} \omega_{2n-1} \,.$$

The integration is over a (2n-1)-dimensional manifold $\mathcal{M}$ with boundary. Please note that, although it is nonlocal in the (2n-2)-dimensional physical space, it is perfectly local in (2n-1) dimensions. From this effective action, the resulting anomaly is

$$\int_{\partial\mathcal{M}_{2n-2}} \epsilon^a \mathcal{A}_a[A] = \int_{\partial\mathcal{M}_{2n-2}} \alpha_{2n-2} \,.$$

While it satisfies the Wess-Zumino consistency conditions because of being the variation of a functional, it is a nontrivial solution thanks to the nonlocality of the effective action. This nice recursive structure that allows us to obtain all the necessary quantities from the anomaly polynomial is known as the **descent equations** [134, 135, 158].

Both α and ω are defined modulo an exact form

$$\begin{aligned} \alpha_{2n-2} &\to \alpha_{2n-2} + d\beta_{2n-3} \,, \\ \omega_{2n-1} &\to \omega_{2n-1} + d\gamma_{2n-2} \,, \end{aligned}$$

because the exterior differential is nilpotent. Then, the gauge variation of ω inherits this ambiguity

$$\delta_\epsilon \omega_{2n-1} \to \delta_\epsilon \omega_{2n-1} + \delta_\epsilon d\gamma_{2n-2} = \delta_\epsilon \omega_{2n-1} + d\delta_\epsilon \gamma_{2n-2} \,,$$

and due to (2.10), it can be seen that the anomaly is thus also defined modulo addition of the gauge variation of a (2n-2)-form

$$\alpha_{2n-2} \to \alpha_{2n-2} + d\beta_{2n-3} + \delta_\epsilon \gamma_{2n-2} \,.$$

It is important to note that, in spite of these ambiguities, the invariant polynomial is unique. In the same way that the freedom to choose γ_{2n-2} does not alter ω_{2n-1}, the choices of both β_{2n-3} and γ_{2n-2} do not change $P(F)$. In other terms, different forms of the anomaly actually refer to one particular anomaly polynomial. In Section 2.1.7, we will comment more on this and try to make a connection to the different physically reasonable choices of the chiral anomaly we proposed above.

Here, we have included only cases where there is a single gauge symmetry. However, generalizations to more complicated polynomials involving several different field strengths can be easily done. The resulting anomalies are called **mixed anomalies**, like the gravitational contribution to the chiral anomaly from (2.9). In fact, this mathematical structure constraints the appearance of anomalies, imposing properties for which we only had physical explanations. The simplest example is that chiral anomalies only appear in even dimensions. What is usually explained because those are the dimensions in which the notion of chirality is defined, it can be here understood because the field strengths in the polynomial are 2-forms. Another property that can be inferred from the descent equations is that pure gravitational anomalies only appear for $4k+2$ dimensions. What is usually explained because CPT reverses the helicity of the fermions in $4k$ but preserves it in $4k+2$, it can be seen here because the Riemann tensors always appear in pairs in the invariant polynomials. Therefore, pure gravitational anomaly polynomials only appear for $D = 4k$.

The discussion about the descent equations appears usually in the literature re-expressed using the **BRST formalism** [22, 21, 20, 143]. Essentially using general properties of the BRST operator s like nilpotency and the fact that it anticommutes with the exterior derivative, it can easily be seen that the Wess-Zumino consistency conditions reduce in this language to the vanishing of the BRST operator acting on the anomaly. In this context, the descent equations are obtained through extensive use of the so-called Russian formula, that states that the field strength remains the same under the following substitution: the exchange of the exterior derivative d by $d+s$ and the addition to the gauge field of the Faddeev-Popov ghost that is introduced in the definition of the BRST operator.

2.1.7 Consistent and covariant currents

The anomaly that can be obtained through the descent equations is called **consistent anomaly** because it satisfies the consistency conditions. We can therefore define the **consistent current** as the current whose divergence is equal to the consistent anomaly. However, in general this form of the anomaly, and subsequently of the current, is not gauge covariant.

It can be shown that there exists a term that can be added to the consistent current that makes the associated anomaly be gauge covariant [19]. This term receives the name of **Bardeen-Zumino polynomial**. The new form of the current after adding this polynomial to the consistent current is called **covariant current** and the quantity equal to its divergence receives the name of **covariant anomaly**. It is indeed gauge covariant. Contrary to the consistent current, that is inserted as an operator through functional variations with respect to the gauge field, this current cannot be obtained through variations of the action and it will only couple to matter covariantly.

Moreover, one could even define from the anomalies a current with no apparent anomaly, whose divergence is zero. We call this current a **conserved current**. Probably it is not surprising at this point that this current is neither the variation of an action nor covariant.

In the abelian case, the difference between the consistent and covariant anomaly amounts only to a factor of three. For the non abelian case, though, the difference is more evident. While the covariant anomaly can be written as the exterior derivative of the Chern-Simons form and therefore it is only a function of the field strengths, the consistent anomaly cannot be written solely in terms of gauge covariant quantities.

The fact that we already introduced a consistent anomaly when discussing the triangle diagrams might be intriguing. Actually, the way the different versions of the anomaly first appeared in this text was through different choices in the labeling of the loop momenta. In particular, the consistent anomaly appeared imposing Bose symmetry. However, as counterintuitive as it might seem at first thought, this makes perfect sense. The consistent current is inserted from variations of the effective action and one expects that the triple variation with respect to the gauge field is symmetric under the exchange of the different insertions.

In the perturbative computation there were two kinds of ambiguities. We can try to make a connection between these ambiguities and the ones faced in the construction of the anomaly from α_{2n-2}. The first kind of ambiguity, associated to the choice of momenta inside the loop, was exploited to construct the meaningful consistent and covariant currents, but it is actually related to β_{2n-3} and the fact that the anomaly is defined up to exterior derivatives. From the construction using the descent equations, it becomes obvious that the Bardeen-Zumino polynomial is actually one particular choice of β_{2n-3} but there are infinitely many.

In an analogous way, the Bardeen counterterm is associated to one particular choice of γ_{2n-2}. The standard use of the Bardeen counterterm is to move the anomaly around the different gauge sectors involved. In particular, we used it to preserve the vector symmetry instead of the axial one, or to preserve diffeomorphisms instead of axial gauge symmetries. But one could in principle use the choice of γ_{2n-2} in as many ways as desired.

By this time, our choice of notation for the currents has probably become obvious. However, since we will make extensive use of it in Chapter 3, let us stop here and discuss it. At the classical level, there is only one kind of current, the Noether current. Therefore, we have chosen to call these currents J^μ, supplemented with subscripts in those cases in which there were several present. However, at the quantum level one can use different definitions of the currents. In particular, along the text we have used calligraphic letters for the consistent currents $\mathcal{J}^\mu$ and we have also used J^μ for the covariant current. There are several equations in which we abused notation, though, by using J^μ inside expectation values and correlators without referring to the covariant current. It was done in those cases in which we had not yet decided one particular quantum current definition and we wanted to refer to the quantum mechanical computation of the classical conservation equations.

2.2 Relativistic hydrodynamics

Hydrodynamics allows us to study the behavior of fluids. In more concrete terms, it can be understood as a low energy effective theory that is valid when the length scales of departure from equilibrium are large compared to the mean free path. It therefore allows for an expansion in small momenta such that there will be a hierarchy in the different contributions depending on the number of derivatives they involve. However, it is not an effective theory in the usual sense because there is no such thing as a well-established effective action, even if this is a very active field of research.

The relativistic form of hydrodynamics applies when the velocities involved are large. Its formalism and equations were thought to be well-known by the middle of past century [47, 101], but the results about anomalous transport have changed our considerations. In the next sections we will concentrate on relativistic hydrodynamics (see [98, 125] for reviews). Particularly, we will cover how the conserved currents can be defined in terms of the hydrodynamic variables and which are the equations of motion associated to them, how the second law of thermodynamics arises in our formalism and, finally, how this formalism is modified by anomalies.

2.2.1 Conserved currents

In the hydrodynamic regime, there is a local notion of equilibrium. It is a good approximation to assume that there is a region around any point that is sufficiently close to equilibrium. This means that one can define for each of those points a temperature T and a chemical potential μ that will satisfy the usual thermodynamic relations. If we want to understand the system we need to try to monitor the conserved quantities associated to these thermodynamic parameters.

Our formalism must be invariant under the action of the Poincaré group, since we are interested in relativistic hydrodynamics. As a consequence, the conserved quantities associated to the thermodynamic parameters appear now in the form of conserved currents. It is possible to match them to the Noether currents if one knows the field theory describing the microscopics of the system. In this context, the conservation equations of the currents play the role of equations of motion.

We will consider in the rest of the section the simple case in which only the energy-momentum tensor $T^{\mu\nu}$ and a $U(1)$ current J^μ appear. The **stress tensor** $T^{\mu\nu}$ can be understood as the conserved current associated to spacetime translations and its conservation law reads

$$\partial_\mu T^{\mu\nu} = 0\,.$$

It is symmetric, a property that is automatically satisfied when we define it as the variation of the action with respect to the metric, and the conserved currents corresponding to the remaining symmetries of the Poincaré group, $\mathcal{M}^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$, are trivially conserved from the symmetry of the energy-momentum tensor. Therefore, they do not give any information and we can safely drop them from our discussion.

The other **conserved current** J^μ also satisfies a conservation law

$$\partial_\mu J^\mu = 0\,.$$

In hydrodynamics it is usually assumed that the theory is parity-symmetric and J^μ is a vector current, with parity-even charge density. However, we could in principle be interested in an axial current or consider a microscopic theory that breaks parity. As a consequence, some terms that are not usually considered in the literature could appear. This will be the case of those associated to anomalous transport. However, we first cover the construction of $T^{\mu\nu}$ and J^μ in a parity invariant theory for pedagogical reasons and only consider the possibility of including parity odd terms in the section about anomalous hydrodynamics.

We can check the number of degrees of freedom, which is independent of the specific structure of the conserved currents. For d spatial dimensions, conservation of the energy-momentum tensor gives $d+1$ equations and conservation of each vector current gives one equation. For our case with only one current it is thus enough to use $d+2$ quantities. On the contrary, if one looks at the energy-momentum tensor and the current, they have $(d+1)(d+2)/2$ and $(d+1)$ independent components, respectively. Thus, we need a prescription to parametrize the conserved currents in terms of the $d+2$ variables and to find a meaningful set of the latter. The usual choice is a local temperature $T(x)$, a local chemical potential $\mu(x)$ and a local fluid velocity $\vec{v}(x)$, but one might take a different set of quantities. We stick to this choice in this introduction.

2.2.2 Constitutive relations

We call **constitutive relations** the resulting form of $T^{\mu\nu}$ and J^{μ} after expressing all the coefficients in terms of T, μ and $\vec{v}$. Before moving on to finding the constitutive relations let us make a remark. In all the expressions from now on, and also on the ones given above for the conservation equations, we will assume a flat background and include partial derivatives. The extension to curved spacetime should be straightforward by taking covariant derivatives instead. Since the covariant derivative of a scalar is equal to its partial derivative, in practice there will only be changes when the differentiation acts on the velocity.

Let us use Lorentz invariance to propose a general definition of the energy-momentum tensor and the current only according to the possible tensor structures. We can introduce a timelike velocity four-vector u^{μ} and decompose the energy-momentum tensor and the current into transverse and longitudinal components using the projector $P^{\mu\nu} \equiv g^{\mu\nu} + u^{\mu}u^{\nu}$. The components of this four-vector in natural units are $u^{\mu} = \gamma(1, \vec{v})$, where $\gamma = 1/\sqrt{1-|\vec{v}|^2}$ is the Lorentz factor. The decomposition of energy-momentum tensor and current in their most general form reads

$$T^{\mu\nu} = \mathcal{E}u^{\mu}u^{\nu} + \mathcal{P}P^{\mu\nu} + q^{\mu}u^{\nu} + q^{\nu}u^{\mu} + t^{\mu\nu}\,, \tag{2.11}$$
$$J^{\mu} = \mathcal{Q}u^{\mu} + \nu^{\mu}\,. \tag{2.12}$$

The coefficients $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{Q}$ are scalars, the vectors q^{μ} and ν^{μ} are transverse to u_{μ} and the tensor $t^{\mu\nu}$ is transverse, symmetric and traceless. We could always recover the coefficients using $u_{\mu}u_{\nu}$ and $P_{\mu\nu}$ as projectors in the longitudinal and transverse direction, respectively.

We already advanced the existence of a derivative expansion due to the deviations from equilibrium being small. In particular, it consists on a expansion in powers of derivatives of the hydrodynamic variables T, μ and u_{μ}, in which ideal fluids are represented by zeroth order terms and any contribution in higher orders will be subleading. For example, viscous and dissipative terms appear all at first order, and so does anomalous transport. If we try to build the coefficients in (2.11) and (2.12) in terms of the hydrodynamic variables, it can be seen that q^{μ}, ν^{μ} and $t^{\mu\nu}$ necessarily contain derivatives. Therefore, ideal hydrodynamics can be fully expressed in terms of the scalar coefficients $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{Q}$.

For static equilibrium, we find $T^{\mu\nu} = \mathrm{diag}(\epsilon, p, p, p)$ and $J^{\mu} = (\rho, \vec{0})$, where ϵ, p and ρ are equilibrium energy density, pressure and charge density, respectively. After performing the Lorentz transformation corresponding to a constant velocity

u^μ and promoting the variables involved (ϵ, p, ρ and u^μ) to slowly varying fields, it can be seen that the scalar coefficients in the constitutive relations can be identified for ideal fluids with a local version of the thermodynamic variables. $\mathcal{E}$ is the local energy density, $\mathcal{P}$ is the local pressure, $\mathcal{Q}$ is the local charge density and u^μ is the local fluid velocity. Furthermore, we can use the appropriate equation of state $p(T,\mu)$ to obtain the first law of thermodynamics $\epsilon = -p + Ts + \mu\rho$ for these variables, where $s = \partial p/\partial T$ is the entropy and $\rho = \partial p/\partial \mu$. Moreover, thanks to the introduction of the entropy in this way, we can also find a second law of thermodynamics in this language. In particular, one can rearrange the longitudinal component of the energy-momentum conservation and the current conservation with the help of the thermodynamic relations to give

$$\partial_\mu (s u^\mu) = 0\,. \tag{2.13}$$

This equation must be interpreted as the conservation of the **entropy current** i.e. the entropy does not increase or decrease in ideal hydrodynamics. However, there could be in principle other zeroth-order terms that produced the non-conservation.

Terms with one derivative have to be taken into account next in the low momentum expansion, but there is an ambiguity that does not appear in ideal hydrodynamics. When one defines the hydrodynamic variables out of equilibrium as fields, there could be many out of equilibrium definitions of the variables that give the same value in equilibrium and only differ from each other by gradients. For example, one could write $\mathcal{E} = \epsilon(T,\mu) + f_{\mathcal{E}}(\partial T, \partial\mu, \partial u)$, where ϵ is determined by the equation of state in equilibrium and $f_{\mathcal{E}}$ represents the gradient corrections that depend on the local temperature, local chemical potential and local velocity. This is a consequence of the fact that there is no operator whose expectation value gives the local value of any of the hydrodynamic variables, as these have no first-principles microscopic definition and are not well-defined out of equilibrium. However, $T^{\mu\nu}(x)$ and $J^\mu(x)$ do have a microscopic definition. Thus, the different definitions of the hydrodynamic variables must be understood as all the possible parametrizations of the conserved currents. The only important thing that all those different redefinitions of the parameters have to satisfy is that the stress tensor and the current remain unchanged. The choice of parametrization is called **frame** in hydrodynamics.

In particular, one could make a frame transformation

$$T(x) \to T(x) + \delta T(x)\,, \quad \mu(x) \to \mu(x) + \delta\mu(x)\,, \quad u^\mu(x) \to u^\mu(x) + \delta u^\mu(x)\,,$$

where δT, $\delta\mu$ and δu are first order in derivatives. If one imposes for this redefinition that $T^{\mu\nu}$ and J^μ remain constant, the only coefficients of the constitutive relations that are not invariant under the redefinition are q^μ and ν^μ, and their change is proportional to δu^μ.

This has two very important consequences. First of all, we can choose the frame to our convenience from the definition of the velocity. Two very usual choices are those for which either ν^μ or q^μ vanishes, and they are called Eckart and Landau frames, respectively. The former implies no charge flow in the local rest frame of the fluid and the latter implies no energy flow. A further option is the frame that we will use in the rest of the thesis, which is the laboratory frame. It represents how the motion of a fluid is seen by an observer that does not move alongside the fluid.

The other consequence is that, since the scalar coefficients are invariant under the redefinition, we can easily represent the transformed gradient corrections in terms of the original ones and choose an off-equilibrium definition of T and μ such that two out of the three scalar coefficients reduce directly to the thermodynamic variables. In the literature it is common to fix $\mathcal{E} = \epsilon$ and $\mathcal{Q} = \rho$ and keep $\mathcal{P}$ unfixed yet. Others just look for combinations that are independent of the chosen frame, but this will not be our approach.

As a result, only determining $\mathcal{P}$, $t^{\mu\nu}$ and a certain combination of q^μ and ν^μ is left to obtain the constitutive relations for first-order hydrodynamics. However, we can use Lorentz symmetry to constraint the possible tensor structures that involve one derivative while being scalars, symmetric transverse traceless tensors or transverse vectors, and then impose the hydrodynamic equations to find the constitutive relations. It can be seen, just as a consequence of Lorentz covariance, that in the constitutive relations there will appear one free coefficient for the scalars, two for each vector and one for the tensor. With this, the equation of state $p(T, \mu)$ can give an expression for ϵ and ρ in terms of T and μ, and the rest of the coefficients in the constitutive relations can eventually be computed as correlators of the current and energy-momentum tensor using linear response theory, which we review in the next section. We will discuss in that section that the physics described by those transport coefficients must be frame invariant and, therefore, the choice of frame doesn't change their value but only the place where they appear in the constitutive relations.

For any order in the derivative expansion, the number of non-vanishing transport coefficients is not as large as the number of independent tensor structures that one can write down consistent with Lorentz symmetry. Therefore, further physical conditions are necessary. In particular, a local form of the second law of thermodynamics helps reducing the number of independent vector structures by one. For a thermal equilibrium state with constant velocity, the entropy current is $S^\mu = su^\mu$ and satisfies (2.13) at zeroth order. However, this entropy current should also have gradient corrections that are functions of the hydrodynamic variables and vanish in equilibrium, giving as a result a local version of the second law of thermodynamics

$$\partial_\mu S^\mu \geq 0 \,.$$

This entropy current is not uniquely defined, but the positivity of entropy production constraints the transport coefficients.

We can use the thermodynamic relation $Ts = p + \epsilon - \mu\rho$ in its covariant form,

$$TS^\mu = pu^\mu - T^{\mu\nu}u_\nu - \mu J^\mu \,,$$

to get a form for the entropy current in terms of the coefficients we have been discussing

$$S^\mu = \left[s + \frac{1}{T}(\mathcal{E} - \epsilon) - \frac{\mu}{T}(\mathcal{Q} - \rho) \right] u^\mu + \frac{1}{T} q^\mu - \frac{\mu}{T} \nu^\mu \,,$$

which is quite easily proved to be frame invariant. It is not surprising now that frames with $\mathcal{E} = \epsilon$, $\mathcal{Q} = \rho$ and either ν^μ or q^μ equal to zero are especially convenient, since this expression simplifies a lot.

2.2.3 Anomalous hydrodynamics

We have already covered some generalities about relativistic hydrodynamics but we have not yet said anything about how hydrodynamics and chiral anomalies combine to give **anomalous transport phenomena**, like the well-known chiral magnetic and chiral vortical effects. These contributions have not been considered in the previous section because we were assuming we were working with a vector current in a parity symmetric theory. However, terms proportional to the vorticity and the magnetic field are not only possible but required for a parity-breaking theory, and are of the same order of dissipative terms.

One could for example focus on a four-dimensional fermion theory that suffers from a gauge anomaly and a mixed gauge-gravitational anomaly. This would in general mean anomalous terms that depend on the connections would arise in the conservation equations of hydrodynamics

$$\partial_\mu J^\mu = P_A(A,\Gamma), \qquad \partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda + Q_A(A,\Gamma).$$

Please note that these expressions must be understood as vacuum expectation values in the presence of quantum contributions like the anomaly. In particular, we are here considering the possibility that an anomaly given by P_A breaks gauge symmetry at the quantum level and an anomaly given by Q_A breaks diffeomorphisms. Furthermore, we make use of the covariant form of the currents because they are good observables, precisely due to their covariance.

For convenience, we consider an effective action for which diffeomorphisms are conserved, i.e. $Q_A(A,\Gamma)$ vanishes. According to what we discussed in Section 2.1, with only one external gauge field present the covariant anomaly for the chiral current reads

$$\partial_\mu J^\mu = -\kappa \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} - \lambda \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left(\mathbf{R}_{\mu\nu} \mathbf{R}_{\rho\sigma}\right) .$$

However, the last term must be dropped in first-order hydrodynamics because it involves two more derivatives than the rest of the contribution, amounting in total to four derivatives.

At this point, one could postulate the possibility that terms proportional to the vorticity and the magnetic field appear in the constitutive relations, as part of q^μ and ν^μ. In particular, if we consider the energy current $J^\mu_\epsilon = T^\mu{}_0$, then

$$\begin{aligned} J^\mu &= \sigma_B B^\mu + \sigma_V \omega^\mu + \dots , \\ J^\mu_\epsilon &= \xi_B B^\mu + \xi_V \omega^\mu + \dots . \end{aligned}$$

The magnetic field and the vorticity, given respectively by

$$\begin{aligned} B^\mu &= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma} , \\ \omega^\mu &= -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma , \end{aligned}$$

involve one derivative, so they appear at the same order as viscous and dissipative terms. There could also be an Ohmic contribution in the constitutive relations, proportional to $E^\mu = F^{\mu\nu} u_\nu$ and thus also first order in derivatives. However, it does not have an anomalous origin so we do not consider it here.

The transport coefficients associated to the magnetic field and the vorticity are even under time reversal. This can be seen from the fact that charge and energy currents, magnetic field and vorticity are all odd under time reversal. As a result, these transport phenomena do not contribute to the entropy production and are **dissipationless**. We will come back to this in the next section about linear response theory.

The specific values of the coefficients $\sigma_{B,\omega}$ and $\xi_{B,\omega}$ depend on the frame choice. In this work we compute the one-point functions in the laboratory frame. However, it was shown by [130] for the Landau frame that most of the unknown coefficients above are uniquely fixed by demanding consistency of the hydrodynamic expansion in terms of the chiral anomaly coefficient κ. In particular, they could calculate those coefficients by imposing the satisfaction of the local form of the second law of thermodynamics. The anomaly produces a new contribution to the divergence of the entropy current of the form $-C\frac{\mu}{T}E \cdot B$ that could have either sign. This is not the right behavior, since the entropy current must grow. Therefore, they include terms proportional to the vorticity and the magnetic field in all the possible vectorial degrees of freedom and use them to prevent the divergence of the entropy current from a negative value.

For our choice of frame [116, 132], the coefficients read

$$\sigma_B = -8\kappa\mu\,, \quad \sigma_V = 2\xi_B = -\left(8\kappa\mu^2 + \gamma T^2\right)\,, \quad \xi_V = -\left(\frac{16}{3}\kappa\mu^3 + 2\mu\gamma T^2\right)\,.$$

The undetermined coefficient γ was linked in [107] to the mixed gauge gravitational anomaly for massless fermions

$$\gamma = 64\pi^2\lambda\,.$$

Therefore, terms including only dependence on μ are coming from the chiral anomaly, while the ones containing T^2 are coming from the gravitational anomaly. All the results have also been confirmed at strong coupling through holographic methods [119, 17, 49, 124, 155, 108, 79, 91, 67, 89, 133]. The universality of the contributions from the gravitational anomaly in the hydrodynamic expansion is somewhat surprising, since the gravitational anomaly itself only enters at third order in derivatives. In fact, strictly speaking the mixed anomaly is not part of the conservation equation of the current in first-order hydrodynamics, as we discussed above.

An interesting feature of these transport phenomena is that they are related to the anomalies through the anomaly coefficient. Therefore, they do not require the anomaly to be switched on. For the mixed anomaly this might seem obvious, because we are seeing that the anomaly itself is of higher order in derivatives and it still has a contribution, but it is also true for pure gauge anomalies. In particular, the appearance of the coefficients proportional to κ does not require the appearance of parallel electric and magnetic fields simultaneously. As long as the anomaly coefficients are nonzero, there is anomalous transport.

2.3 Linear response theory

A systematic way of monitoring how a system reacts to changes in its environment has to be found in order to study transport phenomena. The tool we have at hand for this purpose is called **linear response theory** and we introduce in this section some of the basic ingredients that define it.

The modeling of the outside influence is different for classical and quantum systems. In classical contexts, the changes in the outside are described by forces that are defined as those terms in the equations of motion that are not dynamical and we can know at all times. In principle, these equations of motion do not need to be derived from a Hamiltonian and they can include friction.

However, in quantum mechanical cases, like the ones we are interested in, the outside influence is described by a new term in the Hamiltonian

$$\delta H(t) = \phi_i(t)\mathcal{O}_i(t)\,, \tag{2.14}$$

where there is an implicit sum in the index i. The analogous part to the force here is the **source** ϕ_i, which we can tune as we please, and linear response theory allows us to model how the changes in this quantity alter the different **observables** $\mathcal{O}_i$. For simplicity we will make all the discussion of linear response theory in the context of quantum mechanics and only in the end we will turn to quantum field theory to make the connection with the rest of the work.

If we wanted to know in general how the observables change under the influence of the sources, we would have to solve the full theory. However, we can simplify the problem by assuming that the changes in the source are small and therefore the variation on the one point functions of the observables is linear

$$\delta\langle\mathcal{O}_i(t)\rangle = \int dt'\chi_{ij}(t,t')\phi_j(t')\,. \tag{2.15}$$

Rigorously, this has to be understood as the leading term from a Volterra series in small ϕ_i. This equation serves as the definition of the **response function** $\chi_{ij}(t,t')$.

From now on, we will mostly work in frequency space, so let us include for completeness the definition of the Fourier transform

$$f(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} f(t)\,, \quad f(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega)\,. \tag{2.16}$$

If we assume that the response function is invariant under time translations, i.e. only depends on time intervals and not time itself ($\chi_{ij}(t,t') = \chi_{ij}(t-t')$), it can be seen that the response gives

$$\delta\langle\mathcal{O}_i(\omega)\rangle = \chi_{ij}(\omega)\phi_j(\omega)\,,$$

which means that different frequencies do not mix. This feature would be spoiled if we also considered higher order corrections beyond linear regime. However, in this section we restrict ourselves to the study of linear response functions that are invariant under time translations, and we will use the notation $\chi(t) \equiv \chi(t,0)$.

Let us consider for simplicity that there is only one source in the perturbation of the Hamiltonian. As a result, we can drop all indexes and it is easier to analyze the mathematical properties of the response function, which can unveil in some cases very interesting physical properties too. For example, if the source is real and the associated observable is a Hermitian operator, i.e. it has real eigenvalues, then we know from (2.15) that $\chi(t)$ must be real too. Furthermore, due to the properties of the Fourier transform, this means that

$$\chi(-\omega) = \chi^*(\omega)\,. \tag{2.17}$$

Another interesting property is that the real and imaginary parts of the response function contain different physical information. They can be computed as

$$\begin{aligned}\mathrm{Re}\chi(\omega) &= \frac{1}{2}\left(\chi(\omega)+\chi^*(\omega)\right) = \frac{1}{2}\left(\chi(\omega)+\chi(-\omega)\right)\,,\\ i\mathrm{Im}\chi(\omega) &= \frac{1}{2}\left(\chi(\omega)-\chi^*(\omega)\right) = \frac{1}{2}\left(\chi(\omega)-\chi(-\omega)\right)\,,\end{aligned}$$

where we have used (2.17) to get to the last expressions. If we manipulate them

$$\begin{aligned}\mathrm{Re}\chi(\omega) &= \frac{1}{2}\int dt\left(e^{i\omega t}\chi(t)+e^{i(-\omega)t}\chi(t)\right) = \frac{1}{2}\int dt e^{i\omega t}\left(\chi(t)+\chi(-t)\right)\,,\\ i\mathrm{Im}\chi(\omega) &= \frac{1}{2}\int dt\left(e^{i\omega t}\chi(t)-e^{i(-\omega)t}\chi(t)\right) = \frac{1}{2}\int dt e^{i\omega t}\left(\chi(t)-\chi(-t)\right)\,,\end{aligned}$$

the first thing that becomes evident with these expressions is that the real part is an even function in ω and the imaginary part is odd. In addition, the imaginary part treats differently the real space response function depending on the sign of time. Therefore, it is not invariant under time reversal and it must contain information about dissipative processes. On the contrary, the real part is time reversal invariant. This is the reason why the real and imaginary part sometimes receive the name of reactive and dissipative, or absorptive, part, respectively. Please notice that we discussed above on the nondissipative nature of anomalous transport from the fact that the transport coefficients were even under time reversal. They would thus appear on the real part.

In this context of linear response theory, causality reduces to the vanishing of the response function $\chi(t)$ for negative times. This condition has interesting implications for the response function in frequency space. If we consider the definition of the inverse transform from (2.16), it becomes evident that all the frequencies have a contribution for each value of time t. In order to perform the integral of the inverse transform, we can analytically continue it to the complex plane and the integration can be done by closing the contour with a semicircle of infinite radius in the upper half-plane. The choice of the upper half-plane stems from the fact that the result must vanish for negative time:

$$\lim_{|\omega|\to\infty} e^{-i\left(|\omega|e^{i\theta}\right)(-|t|)} = \lim_{|\omega|\to\infty} e^{-\sin\theta|\omega||t|+i\cos\theta|\omega||t|} = 0\,,\ \text{for } \theta\in(0,\pi)\,.$$

As a result, we can infer using Cauchy's residue theorem that the response function is analytic in the upper half-plane, and this property originally stems from imposing causality.

There are further relations between the real and imaginary part of the response function, which can be obtained from these properties and some basic complex analysis. We will skip the derivation here, though. They are called **Kramers-Kronig relations** and they read

$$\mathrm{Re}\chi(\omega) = \mathcal{P}\int_{-\infty}^{+\infty} \frac{d\omega'}{\pi}\frac{\mathrm{Im}\chi(\omega')}{\omega'-\omega},$$
$$\mathrm{Im}\chi(\omega) = \mathcal{P}\int_{-\infty}^{+\infty} \frac{d\omega'}{\pi}\frac{-\mathrm{Re}\chi(\omega')}{\omega'-\omega},$$

where $\mathcal{P}$ stands for the principal value. There is another version of the relations that shows how the dissipative part contains all the information about the response function

$$\chi(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega'}{\pi}\frac{\mathrm{Im}\chi(\omega')}{\omega'-\omega-i\epsilon}.$$

Let us now go back to the perturbed Hamiltonian (2.14). The full Hamiltonian reads

$$H = H_0 + \delta H.$$

We want to treat the problem in the interaction picture. Thus, we need to define a unitary time evolution operator

$$U(t,t_0) = Te^{-i\int_{t_0}^{t}\delta H(t')dt'},$$

which will satisfy by construction Schrödinger's equation with the associated Hamiltonian. This operator will give the time evolution of the states

$$|\psi(t)\rangle_I = U(t,t_0)|\psi(t_0)\rangle_I,$$

and of the density matrix

$$\rho_I(t) = U(t,t_0)\rho_I(t_0)U^{-1}(t,t_0). \tag{2.18}$$

if the states are described by it. We are implicitly assuming in all this discussion that $t_0 \to -\infty$ and the interaction was not switched on at that time. From now on, we will drop the subscript I denoting we are in the interaction picture .

Using (2.18), we express the one-point function of any operator as

$$\langle \mathcal{O}_i(t)\rangle = \mathrm{Tr}\left(\rho(t)\mathcal{O}_i(t)\right) = \mathrm{Tr}\left(\rho(t_0)U^{-1}(t,t_0)\mathcal{O}_i(t)U(t,t_0)\right).$$

It can be expanded perturbatively in small source

$$\langle \mathcal{O}_i(t)\rangle = \mathrm{Tr}\left(\rho(t_0)\left(\mathcal{O}_i(t) + i\int_{t_0}^{t} dt'\left[\delta H(t'),\mathcal{O}_i(t)\right] + \dots\right)\right),$$
$$= \langle \mathcal{O}_i(t)\rangle_0 + i\int_{t_0}^{t} dt'\left\langle\left[\delta H(t'),\mathcal{O}_i(t)\right]\right\rangle_0 + \dots,$$

where the notation $\langle \ldots \rangle_0$ means an equilibrium average with the unperturbed Hamiltonian H_0. If we now drop the higher terms in the ellipsis and we extend the integration from t to $+\infty$ with the use of a Heaviside function, we can compare this equation to (2.15) and obtain the response function in a quantum theory

$$\chi_{ij}(t-t') = -i\theta(t-t')\left\langle \left[\mathcal{O}_i(t), \mathcal{O}_j(t')\right] \right\rangle_0 .$$

This result is known as the **Kubo formula** [100]. The computation of the response thus reduces to the computation of this correlator.

2.3.1 Transport coefficients revisited

Once we have introduced the Kubo formalism, our aim is to use linear response theory to compute transport properties in quantum field theory, so we generalize all the quantities to not only depend on time but also on space. The generalization is sort of trivial, changing the arguments t by $x \equiv (t, \vec{x})$ and ω by $k \equiv (\omega, \vec{k})$ except for the step function in the Kubo formula. One can easily recognize that all the correlators now turn into Green's functions, as it is usual for quantum field theory, and, in particular, the one in the Kubo formula is a **retarded Green's function**.

Let us now consider a free theory of a boson (or a fermion) that couples to a external gauge field

$$\delta H = A_\mu J^\mu .$$

This will help us show how to compute transport coefficients in a simple setup. Applying the Kubo formula, we could monitor the changes in the current due to variations of the source as

$$\delta\langle J^\mu \rangle = -i \int_{-\infty}^{+\infty} d^4x' \theta(t-t') \left\langle \left[J^\mu(x), J^\nu(x')\right] \right\rangle_0 A_\nu(x') .$$

Rewriting this expression analogously to (2.15) and Fourier transforming it, we obtain

$$\begin{aligned}
\langle J^\mu(k) \rangle - \langle J^\mu(k) \rangle_0 &= \int_{-\infty}^{+\infty} d^4x \int_{-\infty}^{+\infty} d^4x' e^{ik\cdot x} \chi_{\mu\nu}(x-x') A_\nu(x') \\
&= \int_{-\infty}^{+\infty} d^4x \int_{-\infty}^{+\infty} d^4x' e^{ik\cdot(x-x')} \chi_{\mu\nu}(x-x') e^{ik\cdot x'} A_\nu(x') \\
&= \chi_{\mu\nu}(k) A_\nu(k) ,
\end{aligned} \tag{2.19}$$

where we introduce the notation $k \cdot x = \omega t - \vec{k} \cdot \vec{x}$ and the dependence in k simply means that it is a function of $(\omega, \vec{k})$. We are now ready to apply the Kubo formalism to a particular example.

The natural first choice is the response to an electric field, the well-known Ohm's law. In the gauge in which $A_0 = 0$, which simplifies this computation greatly, the electric field is defined in real and momentum space as

$$E_i(x) = -\partial_t A_i(x) , \quad E_i(k) = i\omega A_i(k) .$$

We can suppose the transport coefficient associated to Ohm's law is a generic conductivity $\sigma_{ij}(k)$ and the current then reads

$$\langle J_i(k)\rangle = \sigma_{ij}(k)E_j(k)\,.$$

If we now substitute this in (2.19) and express everything in terms of $A_i(k)$, we find

$$\langle J_i(k)\rangle = \sigma_{ij}(k)i\omega A_j(k) = \langle J_i(k)\rangle_0 + \chi_{ij}(k)A_j(k)\,.$$

Finally, we can obtain an expression of the transport coefficient by functional variation with respect to the source

$$\sigma_{ij}(k) = \frac{1}{i\omega}\left(\chi_{ij}(k) + \int \frac{d^4k'}{(2\pi)^4}\left\langle \frac{\delta J_i(k')}{\delta A_j(k)}\right\rangle_0\right)\,.$$

The second contribution is sometimes called contact term. We have introduced the integrals to drop the Dirac deltas that arise because of functional differentiation. If we wanted the DC limit, it would be safe to take $\vec{k}=0$ in this expression and then carefully take the limit of ω going to zero. In other cases in which the momentum appeared explicitly, like responses to the magnetic field, we would instead have to take the limit for $\vec{k}\to 0$.

At this point one might be tempted to think that the functional derivative with respect to the sources could insert new currents in the correlators. However, these expectation values are computed using the unperturbed Hamiltonian H_0. Therefore, the sources are not present, and no current is inserted. Nevertheless, the expressions of the currents can in principle depend on the sources and that is precisely why contact terms like $\langle \delta\mathcal{O}/\delta\phi\rangle_0$ can sometimes survive.

The expressions of the currents are in some cases more complicated than in this version of the Ohm's law. Imagine, for example, relativistic hydrodynamics [98]. We know how to couple the conserved currents: the energy-momentum tensor couples to the metric $g_{\mu\nu}$, which we take as a perturbation around flat space $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, and the current couples to the gauge field A_μ. We can therefore relate the different transport coefficients to combinations of correlators of these currents by taking perturbations around a certain background an expressing the hydrodynamic variables in terms of the external sources $h_{\mu\nu}$ and A_μ. From a hand waving argument, the conserved currents J^μ and $T^{\mu\nu}$ are inserted in the correlators by variations with respect to A_μ and $h_{\mu\nu}$, respectively.

However, we have seen above there is an ambiguity in the choice of frame in relativistic hydrodynamics, which means there is not only one form for the transport coefficients. In particular, the different definitions of the transport coefficients amount to different combinations of the components of the response function. However, the physics of the system must be the same independently of the description and the result should not depend on the choice of the velocity field. Indeed, the arbitrariness of the flow velocity translates into the possibility to always express the transport coefficients independently of correlators of the velocity. The latter can always be eliminated, and the transport coefficients are expressed in terms of the correlators of the conserved currents only, as discussed in [6].

Regarding linear response computations, there is a simple way of reinterpreting the dependence on the velocity [107]. We can always go to the rest frame of the fluid, which is defined by $u^\mu = (1, 0, 0, 0)$, at the expense of the appearance of a gravitomagnetic potential $h_{0i} = (A_g)_i$ due to the boost. This gives $u_\mu = (1, \vec{A}_g)$ and allows us to write the local fluid velocity as $\vec{v} = \vec{A}_g$. Thus, thanks to this trick all the dependence on the velocity at the constitutive relations can be recasted as dependence on the metric, which does have a conserved current associated to it. As a result, we can easily find the components of the response functions that give the transport coefficients in relativistic hydrodynamics.

Before finishing the section about linear response, we include for completeness the application of the Kubo formalism to the computation of some of the anomalous transport coefficients in the laboratory frame. We will focus on the chiral magnetic and chiral vortical effects on the $U(1)$ current. The constitutive relation for the electromagnetic current reads

$$\left\langle J^i(x) \right\rangle = \sigma_B \epsilon^{ijk} \partial_j A_k(x) + \sigma_V \epsilon^{ijk} \partial_j v_k(x) \, ,$$

where the first transport coefficient σ_B is the one associated to the chiral magnetic effect and σ_V is related to the chiral vortical effect. We first substitute the velocity by the gravitomagnetic field and Fourier transform this expression

$$\left\langle J^i(k) \right\rangle = \sigma_B \epsilon^{ijk} i k_j A_k(k) + \sigma_V \epsilon^{ijk} i k_j h_{0k}(k) \, .$$

Please note that the only sources that we need to consider are A_k and h_{0k}. Using now the Fourier transformed definition of the response function we find another expression for the current

$$\left\langle J^i(k) \right\rangle = \left\langle J^i(k) \right\rangle_0 + \chi_{ij}(k) A_j(k) + \chi_{i,0j} h_{0j}(k) \, ,$$

where

$$\chi_{ij}(k) = -i \int d^4x e^{ik\cdot x} \theta(t) \left\langle \left[J^i(x), J^j(0) \right] \right\rangle_0 \, ,$$
$$\chi_{i,0j}(k) = -i \int d^4x e^{ik\cdot x} \theta(t) \left\langle \left[J^i(x), T^{0j}(0) \right] \right\rangle_0 \, .$$

Equating both expressions we get a relationship among the transport coefficients, response functions and the equilibrium expectation value of the current. In the context of the chiral anomaly, the current J^i is the usual fermion current, so variations of the current with respect to the sources will give no contribution. Therefore, we can safely obtain the transport coefficients taking functional derivatives with respect to the sources. The final result for constant magnetic field and vorticity gives

$$\sigma_B = -\, i \lim_{\vec{k}\to 0} \epsilon^{ijl} \frac{\chi_{ij}(0, \vec{k})}{2k_l} \, ,$$
$$\sigma_V = -\, i \lim_{\vec{k}\to 0} \epsilon^{ijl} \frac{\chi_{i,0j}(0, \vec{k})}{2k_l} \, .$$

With these equations we conclude the review of linear response theory. We will be able to apply it to strong coupling dynamics when we find a way to compute the Green's functions in the context of AdS/CFT. We now move on to cover the basic aspects of the correspondence.

2.4 Holography

More than forty years ago, several authors established what got to be known as **black hole thermodynamics** [30]. Different quantities associated to black holes could be reinterpreted as thermodynamic variables and satisfy the same equations. Among the different new thermodynamic relations that were proposed, one stood as particularly intriguing: the area of the horizon played the role of entropy for the black hole. It was first proposed by Bekenstein, as a proportionality relation. However, Hawking managed to fix the expression to

$$S_{BH} = \frac{k_B A}{4\ell_P^2}$$

in its seminal work about the thermal radiation emitted by black holes that now receives his name [74]. In that paper, one of the major puzzles in modern theoretical physics was also first encountered, the so-called **information paradox**, which is an extremely interesting topic that would require a whole new thesis by itself. The result of the **Bekenstein-Hawking entropy** got a solid confirmation when a string theoretical computation gave the right result for a class of five-dimensional extremal black holes [136]. The result was then reproduced with similar techniques for many other classes of black holes, but such a computation for the Schwarzschild black hole is still missing.

The Bekenstein-Hawking entropy is especially striking because it seems to indicate that all the information about the 3-dimensional black hole is actually stored in its 2-dimensional horizon. It also saturates the **Bekenstein bound**, which was proposed few years later [23] and establishes a maximum of entropy that can be stored in a finite region of space with a finite amount of energy.

This lead to the proposal of the **holographic principle** as a property of string theory and, supposedly, any theory of quantum gravity. According to the principle, the information about any region would be encoded on its lower-dimensional boundary. It was first proposed by 't Hooft [76] and further developed by Susskind in the context of string theory [137], although Charles Thorn had been holding similar ideas for some time [140].

The most prolific realization of the holographic principle so far has been the AdS/CFT correspondence. Its successes are so abundant and profound that it has somehow convinced the high energy physics community that the holographic principle is a property to be expected from any theory of quantum gravity. In what remains of this chapter we will try to cover all the basic necessary aspects of it.

2.4.1 Brief review of the AdS/CFT duality

The **AdS/CFT conjecture** states that conformally invariant theories in d dimensions are equivalent in the large N limit to supergravity on $d+1$ dimensional AdS space times a compact manifold. In those cases in which the compact manifold is a sphere, the dual theory is maximally supersymmetric.

The paradigmatic example is that of 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory, which is dual to Type IIB superstring theory in $AdS_5 \times S_5$. This example was already studied in detail in the original paper of the duality [114]. In fact, one can make an intuitive early check of the duality by matching the symmetries of both theories. First of all, the isometry group of AdS_5 is $SO(4,2)$, which becomes evident if we express AdS as an embedded hyperboloid

$$x_1^2 + x_2^2 + x_3^2 - x_0^2 - x_{-1}^2 = -\alpha^2$$

in $\mathbb{R}^{4,2}$, whose metric is $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2 - dx_{-1}^2$. Regarding S_5, the isometry group is $SO(6)$. Therefore, as a result, the total isometry group is $SO(4,2) \times SO(6)$ on the gravity side. Appropriately, for the CFT the $SO(4,2)$ component comes from the conformal group, and the $SO(6)$ is the internal R-symmetry relating the six scalar fields and four fermions, thus matching the isometry group of the gravitational theory. Moreover, though it is not as simple to show, there is a further matching between the 32 supersymmetries of both sides.

It is also instructive to check how the free parameters of both theories relate to each other. On the gravity side, the relevant dimensionless parameters are the string coupling g_s and the curvature scale in units of the string length L/l_s. There exists a relation to the gauge coupling g_{YM} and the rank of the gauge group N from the CFT, that can be better understood in terms of the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$. It is given by

$$\frac{L^4}{l_s^4} \propto \lambda \,, \tag{2.20}$$

$$g_s \propto \frac{\lambda}{N} \,. \tag{2.21}$$

We must remark that, though most of the community nowadays expects the duality to hold for the whole range of parameters, the limit in which it was found, and the one which we will be using in our work, is that of large λ and even larger N. On one hand, $\lambda \gg 1$ guarantees that L is large in string units and, therefore, there are no stringy corrections to the geometry. On the other hand, $N \gg \lambda \gg 1$ makes the string coupling small and, therefore, the quantum corrections become irrelevant. Furthermore, in this limit it becomes manifest that the AdS/CFT duality is a strong/weak coupling duality: it allows us to study gauge theories at strong coupling via weakly coupled gravity theories.

Some more examples of theories that satisfied the duality were already given in the original paper by Maldacena. Moreover, since the conformal field theory is obtained as the projection at infinity of the bulk gravitational physics, one could obtain more general nonconformal field theories by modifying the boundary conditions from those of AdS. This suggested the existence of a more general gauge/gravity duality and somehow anticipated one of the nicest features of the duality. Holography establishes an equivalence between the holographic direction and an energy scale, thus geometrizing the renormalization group flow in a very natural way. The boundary of AdS would correspond in this picture to the UV of the gauge theory and, consequently, the AdS asymptotics (with its $SO(4,2)$ isometry group) would be associated to an ultraviolet fixed point from which the renormalization group would flow. In Chapter 3 we explore this idea of holographic renormalization group in the context of anomalous transport.

The full computational power of the AdS/CFT duality was uncovered when a prescription to compute correlators of the CFT was finally proposed [153, 64]. One first establishes the equivalence between the partition functions of each side. In fact, in the weak coupling limit in which gravity becomes classical, the partition function of the bulk is essentially the exponential of the on-shell action. In this matching of partition functions, the one associated to the gravity side can be seen as a function of the boundary conditions, much in the same way it is a function of the sources of each operator for the field theory

$$Z_{gravity}\left[\phi = \Phi|_{\partial \mathrm{AdS}}\right] = \left\langle e^{-\int \phi \mathcal{O}} \right\rangle_{\mathrm{CFT}} ,$$

where we denote the bulk fields by Φ and their boundary values by ϕ, to clearly make the connection to the sources of (2.14). The different correlators can be obtained by functional differentiation with respect to the sources and they will then be interpreted as propagation in the bulk of the associated fields with appropriate boundary conditions fixed by the sources. Furthermore, this gives an interpretation to the form of the bulk fields near the boundary.

The boundary of AdS is a regular singular point. This means that it appears as singular in a very particular way at the level of the equations of motion for the bulk, but it has regular asymptotic solutions. Those series solutions around the boundary, obtained using Frobenius method, possess only two independent modes. Therefore, a generic field Φ can be approximated as

$$\Phi = r^{-\Delta_-}\left(\phi + \ldots\right) + r^{-\Delta_+}\left(\hat{\phi} + \ldots\right) .$$

The exponents $\Delta_\pm$ are different depending on the particular field, but we can go back to the equivalence of the partition functions to interpret the different quantities in this asymptotic expansion. In particular, it can be shown that ϕ plays the role of the source, $\hat{\phi}$ is proportional to the expectation value of the operator and Δ_+ is its conformal dimension.

The bulk field corresponding to a certain inserted operator is found using symmetries, since both objects should have the same Lorentz structure and quantum numbers. For example, we can go back to the examples relating linear response theory with anomalous hydrodynamics. External gauge fields are source for currents associated to global symmetries, and they can be seen as the boundary values of dynamical gauge fields in the bulk. Similarly, the projection on the boundary of bulk gravitons is the source for the energy-momentum tensor. Moreover, all this construction can be generalized to cases with multiple operators sourced simultaneously and the mixing of the operators translates to interactions between the fields in the bulk.

So far we have discussed the part of the holographic dictionary that refers to operators, but we have not commented on the states. The two simplest cases are pure AdS, which corresponds to the vacuum of field theory, and an AdS black hole, which is dual to a thermal state. The temperature of the thermal state can be obtained as the Hawking temperature of the black hole and, if there were chemical potentials present, they are computed as the difference in the zero-component of the gauge field between the boundary and the horizon. This computational definition can be better understood if we interpret the chemical potential as the potential that must be overcome to bring a charge from the boundary to the horizon.

In this discussion of the holographic dictionary we have decided not to include many details or mathematical expressions for the sake of brevity. While the duality is the central technique in all of our projects, our true aim is to understand anomalous transport phenomena. Thus, we are not really interested on the correspondence itself. However, we perform many holographic computations in the rest of the chapters, so the practical workings of the duality will still become explicit below. If the reader is interested on finding more details on the theoretical aspects of the duality, she could find them in the plethora of available reviews and books [3, 7, 80, 118].

The gauge/gravity duality has proved itself as a technique of invaluable computational power in contexts where there was no other available tool. In particular, it allows to study the perturbatively inaccessible field theory regime of strong coupling through simpler gravitational computations. Thus, it has somehow brought the long-waited expectation of comparing string theory to experiments. As a consequence, it has been extensively applied to contexts as varied as QCD [31], hydrodynamics [81] or condensed matter physics [156]. In those cases in which it was possible to perform the computations at both sides of the correspondence, the agreement has been perfect. Another of the reasons why application of holography have been so popular is that out of equilibrium physics, which are usually difficult to study, are very simply implemented thanks to the duality, as it will become manifest in Chapter 6: they reduce to time dependence on the gravity side.

There has been some criticism along the years on the extensive applications of holography. Some of the most prominent arguments against them, and the duality in general, are that the correspondence is still a conjecture and not a theorem, that it could not describe the universe because we observe the universe to be dS and not AdS, that super Yang-Mills is not really QCD or even that the nonsupersymmetric bottom-up models usually used in applications to condensed matter theory possessed unstable vacuum solutions [120]. In our projects however our approach has been to assume the duality to be true and perform computations in general relativity in order to gain intuition about general features of the anomaly induced transport phenomena in strong coupling regimes. Furthermore, although we used nonsupersymmetric models in all of our projects, we expect all the presented computations to be reproducible in supersymmetric theories.

2.4.2 Anomalies in holography

Chiral anomalies are included in holography through the addition of **Chern-Simons terms** to the action [49, 17], like

$$S \to S + \int d^{d+1}x\sqrt{-g}I_{CS}\left[A, F, \mathbf{\Gamma}, \mathbf{R}\right] .$$

This must be understood in the context of the anomaly inflow picture, where these Chern-Simons terms, that are local terms in the bulk, play the role of the nonlocal effective action of the dual field thoery. However, what is somehow peculiar in the context of AdS/CFT is that we will be performing all the computations in the bulk and, therefore, those Chern-Simons terms have a contribution in the relevant equations of motion.

All the anomalies in holography are 't Hooft anomalies, as they are associated to global symmetries of the dual field theory. Otherwise, they would have to be canceled for the theory to be consistent. However, it is important to note that they are associated to bulk gauge symmetries and the associated fields are true gauge fields.

We will consider in the following a very rich structure of anomaly effective actions. In general, the possible abelian chiral anomalies in five-dimensional holographic theories can be characterized as: $U(1) \times U(1)^2$ anomalies, like the Adler-Bell-Jackiw anomaly from (2.5); $U(1)^3$ anomalies, which would be the global version of the gauge anomaly from (2.6); and mixed gauge-gravitational anomalies, like the one from (2.9). We now comment on the second and third types, because they illustrate all the interesting aspects.

In a four-dimensional field theory, the bulk Chern-Simons action that gives rise to a pure $U(1)^3$ anomaly and a mixed gauge-gravitational anomaly reads

$$I_{CS} = \epsilon^{\mu\nu\rho\sigma\tau}A_\mu\left(\frac{\kappa}{3}F_{\nu\rho}F_{\sigma\tau} + \lambda\mathrm{Tr}\left(\mathbf{R}_{\nu\rho}\mathbf{R}_{\sigma\tau}\right)\right) , \tag{2.22}$$

where $F_{\mu\nu}$ is the field-strength of A_μ and $\mathbf{R}_{\nu\rho}$ is Riemann's tensor expressed as a 2-form. We have implicitly introduced a notation we will use in the rest of this thesis: boldface is used when referring to forms whose internal indexes are dropped.

The dynamics are not affected by the addition of a total derivative to the action. In particular, we can use this freedom to move the anomaly from one sector to another. We explicitly work it out for the mixed anomaly, since we will later use the form of the anomaly for which diffeomorphisms are anomalous. The total derivative is

$$-4\lambda\nabla_\nu\left(\epsilon^{\mu\nu\rho\sigma\tau}A_\mu\mathrm{Tr}\left(\mathbf{\Gamma}_\rho\partial_\sigma\mathbf{\Gamma}_\tau + \frac{2}{3}\mathbf{\Gamma}_\rho\mathbf{\Gamma}_\sigma\mathbf{\Gamma}_\tau\right)\right) . \tag{2.23}$$

If we add it to (2.22), it gives

$$I'_{CS} = \frac{\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau} + 2\lambda\epsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}\mathrm{Tr}\left(\mathbf{\Gamma}_\rho\partial_\sigma\mathbf{\Gamma}_\tau + \frac{2}{3}\mathbf{\Gamma}_\rho\mathbf{\Gamma}_\sigma\mathbf{\Gamma}_\tau\right)$$

or in terms of the curvatures

$$I'_{CS} = \frac{\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} + \lambda \epsilon^{\mu\nu\rho\sigma\tau} F_{\mu\nu} \mathrm{Tr}\left(\mathbf{\Gamma}_\rho \mathbf{R}_{\sigma\tau} - \frac{1}{3}\mathbf{\Gamma}_\rho \mathbf{\Gamma}_\sigma \mathbf{\Gamma}_\tau\right) . \tag{2.24}$$

This has no effect on the bulk dynamics but it modifies the definition of the current operators. While the physical content of them is the same in both descriptions, their interpretation is slightly different in the two cases.

In holography the notion of consistent and covariant currents is very transparent, and completely analogous to quantum field theory. Chapter 3 is devoted to understand the different definitions of the currents and their use in the context of the holographic renormalization group flow to simplify the computations. Thus, we will not include much detail here, but let us briefly comment on them for the sake of completeness of this introductory chapter. The consistent current in holography is the one that is obtained by projecting at the boundary the variation of the on-shell action with respect to the associated gauge field. The covariant current, on the other hand, is always obtained from the consistent current as the result of subtracting the gauge variation of the consistent current from itself. In general, the redefinitions of the current can be done through the use of counterterms, like the total derivative in (2.23). In fact, this particular one is the extension to holography of a Bardeen counterterm that moves the anomaly from the gauge sector to the diffeomorphism sector.

We conclude the theoretical introduction with this brief review of anomalies in holography. It is now time to move on to the original content of this thesis.

Chapter 3

Holograhic renormalization group approach to anomalous transport

Holographic renormalization group (RG) flows are better understood with the help of radially conserved quantities. Since the radial coordinate is dual to an energy scale, those quantities do not get renormalized. Finding them will allow us to find exact relations between their values at different points along the holographic direction.

We are interested in field theories whose global symmetries are anomalous. This means that their associated bulk gauge invariance will be broken by Chern-Simons terms. However, one could still play with the definition of the currents in order to find quantities with suitable conservation laws. We will introduce then what we call membrane currents, that possess a nice behavior with respect to the RG. Their conservation laws can then be used to characterize the anomalous transport in terms of the currents at the horizon.

In this project we make a special emphasis on gravitational anomalies. They arise due to Chern-Simons terms that are higher order in derivatives and, consequently, the whole construction of the currents becomes more intricate. However, it can be performed successfully and it helps us better understand the anomalous gravitational transport coefficients.

The chapter is based on [36]. It is organized as follows: in Section 3.1 we put our work in context and motivate it; in Section 3.2 we review the construction of the membrane currents and the Wald procedure, and we then apply it to a nonanomalous preliminary example; in Section 3.3 we use the constraint equations of the bulk theory to propose a definition of membrane currents and we compute the radially conserved fluxes for anomalous theories; in Section 3.4, we extract the anomalous transport coefficients from the radially conserved quantities, and, finally, in Section 3.5 we sum up the chapter and include some concluding remarks.

Let us make some comments about notation and conventions before moving on to the actual content of the chapter. Since we will be treating the radial AdS coordinate separately because of its role as an energy scale, we need to perform a $d+1$ decomposition using the ADM formalism [11]. As reviewed for example in [121], this simplifies if we use the Fefferman-Graham gauge for the bulk metric

$$ds^2 = d\hat{r}^2 + \gamma_{ab} dx^a dx^b \,, \tag{3.1}$$

where the asymptotic AdS expansion reads

$$\gamma_{ij} = e^{2\hat{r}}\gamma_{ij}^{(0)} + \gamma_{ij}^{(-2)} + e^{-2\hat{r}}\gamma_{ij}^{(-4)} + \dots .$$

In this expression $\gamma_{ij}^{(0)}$ must be interpreted as the (curved) background metric of the CFT.

The spacetime is subsequently foliated with surfaces Σ of constant $\hat{r}$ and it thus makes the renormalization group interpretation very transparent. This gauge fixing can only be consistently imposed near the boundary and it might produce artificial singularities near the horizon, so we will switch to a slightly different coordinate system in that region. The new radial coordinate R will be implicitly defined through asymptotics in Section 3.4. At some points the ADM decomposition of some of the quantities might not be explicit, but all the results can be recovered by the use of identities $-\Gamma^{\hat{r}}_{ab} = K_{ab}$ and $\Gamma^{b}_{a\hat{r}} = K^{b}_{a}$, where K_{ab} stands for the extrinsic curvature on those hypersurfaces Σ and is given in Fefferman-Graham coordinates by $K_{ab} = \frac{1}{2}\partial_{\hat{r}}\gamma_{ab}$.

We use Greek indexes like μ and ν to denote bulk tensors, while tensors on the boundary Σ are denoted with Latin letters a, b, etc. When confusion may arise, quantities intrinsic to Σ are denoted by hatted names, like $\hat{R}_{abcd}$. This choice of indexes is also used in the rest of the thesis. The bulk covariant derivatives are denoted by ∇_μ and the membrane covariant derivatives by D_a. In much of the formal discussion we employ an on-shell formalism. Then, we denote equality up to equations of motion by the $\doteq$ symbol.

3.1 Motivation

During the last years there has been a renewed interest in the dynamics of gravitational theories on null horizons. The first works date back to the early days of the membrane paradigm, according to which the geometric fluctuations of the null geometry can be seen as hydrodynamic modes through the projection of Einstein's equations [141]. However, in the context of holography this construction receives a new interpretation because the radial coordinate of AdS spacetime can be interpreted as an energy scale for the dual theory [75, 50, 126]. The relationship can only be established explicitly in some particular cases, but they suggest that fluctuations of the horizon geometry could be related to the low energy physics of the dual field theory.

This intriguing feature can only be studied quantitatively if a relation between the one-point functions and the horizon data is known, which typically requires the solution of the bulk dynamical equations. Even then, it is sometimes not straightforward to disentangle the contribution due to the horizon from the rest of the bulk. However, in some special cases the relation can be established and the dual field theory observables are fully expressed in terms of quantities evaluated at the horizon. In [82], the authors considered homogeneous neutral backgrounds and were able to show that some particular combinations of bulk fields were independent of the radial coordinate in the low momenta zero frequency limit, which is also called DC limit. This observation allowed them to compute general responses in that limit in terms of horizon fluctuations, showing perfect agreement with the usual results. According to this technique, there was no necessity to know the dynamics of the rest of the bulk.

Later works have been able to extend this mechanism to more general theories, notably [43, 45, 46]. The most interesting result is the realization that the radially conserved quantities are really the fluxes of certain currents through constant radial coordinate hypersurfaces Σ. The integration over the hypersurfaces in those fluxes explains in a very natural way why the responses could only be computed through this procedure at the DC limit. Besides this, these works were also able to show that this reasoning applies to the $U(1)$ current, but also to the dual heat current if the spacetime was stationary, thus giving a prescription to compute responses in the energy current.

Although the connection was pointed out later [111], the conserved fluxes are closely related to the Komar charges of general relativity. They are rigorously defined by the Wald procedure [148] and this method allows one to obtain a closed $(d-1)$-form k, that does not vanish on-shell, from the variation of the action under gauge transformations or diffeomorphisms. The closure relation of this quantity implies under certain conditions that its fluxes through the hypersurfaces Σ introduced above are conserved radially and those fluxes are related to the fluxes of the membrane currents. In the last years, this link has been used to compute DC responses in many gravitational theories.

As already said, holography has been used extensively to study transport phenomena induced by anomalies [102]. Most work has been devoted to analyzing the DC responses sourced by magnetic field and vorticity in four dimensional field theories that possess an anomalous $U(1)$ symmetry. Holography has been particularly useful linking part of the transport coefficients to the mixed gauge gravitational anomaly [107]. As already discussed, this relation is surprising because the gravitational contribution to the axial anomaly seems to appear at higher order in derivatives, and therefore it would not be expected to contribute at leading order in the hydrodynamic expansion. While the way this result arises in quantum field theory is a little bit obscure, in holography it can be understood as coming from a purely extrinsic term that appears when the five-dimensional action is projected onto the four-dimensional spacetime where the dual theory lives.

This contribution from the gravitational anomaly has received further attention in different contexts. Various arguments have been developed from the point of view of effective field theory to fix the coefficients only from equilibrium considerations [89, 88, 87]. In holography it has been seen it could arise as horizon fluctuations in [33] and the relation to the dual one point function was given from the bulk equations of motion in [13]. Furthermore, it was recently showed by various authors [56, 55] that a part of this contribution can be explained through the matching of the global gravitational anomalies.

In this work we connect the aforementioned two lines of research, i.e. holographic conserved charges and anomalous transport. In order to do so, we need to extend the membrane paradigm to Chern-Simons theories, looking for quantities that satisfy constraints formally equal to the usual field theory Ward identities, and show that these match the conserved quantities given by the fluxes of the Komar charges. The Wald construction becomes complicated for these theories because the

Chern-Simons actions are noncovariant, but we make use of previous work from [138, 29, 12].

Since there have been many similar works in the last years, let us conclude this section with a discussion on how our work relates to the references in the literature:

- The general conservation equations for the $U(1)$ currents were already derived by [65, 60] by direct analysis of the equations of motion, which allowed them to prove the universality of the $U(1)$ transport. When compared to the work presented in this chapter, ours extends the discussion to energy fluctuations and formalizes the connection between the conserved quantities and the anomalous Wald construction. Such an use of the dynamical equations to relate the value of the CFT currents with horizon fluctuations was previously hinted by the work of [13].

- The connection between the conserved fluxes of [43] and the Wald construction is not new. However, to our knowledge it was formalized recently [111], while the work this chapter is based on was being completed. Its application to the anomalous theories had yet to appear. We formalize such extension and identify the conserved fluxes as the suitably defined membrane currents of the theory.

- The constraint equations were already analyzed in the near horizon region in [33]. However, an explicit link to the membrane currents was missing.

3.2 Membrane currents and conserved charges

3.2.1 Membrane currents

We start giving a procedure to obtain membrane $U(1)$ currents and a membrane stress tensor. We can define them as fields living on a spacelike hypersurface which are asked to satisfy relations formally equal to the usual field theory Ward identities. We can resort to the usual constraint equations of the bulk theory to find them as projections of bulk fields onto the hypersurfaces. In principle, they might need to undergo holographic renormalization to avoid singularities. However, the counterterms added do not change the form of the Ward identities or the DC transport properties, so we will no further comment on them.

The constraint equations appear as components in the equations of motion, usually associated to the radial direction, so they are expected to hold all along the bulk. However, they only possess a clear interpretation in the boundary, where they are equivalent to the field theory's Ward identities. We use this intuition to propose that the constraint equations in other points of the foliation must be interpreted as Ward identities in the effective theory description at low energies and exploit it for our purposes.

Let us therefore introduce a hypersurface Σ of constant radial coordinate $\hat{r}$. We consider a generic action containing a metric and a gauge field. Only later we will introduce particular examples. If we want to obtain the constraints for an arbitrary value of $\hat{r}$, we cut the radial integration of the action at the associated Σ. The on-shell variation of the action evaluated at such a hypersurface reads

$$\delta S_\Sigma = \int_\Sigma d^d x \sqrt{-\gamma} \left(\frac{1}{2} t^{ab} \delta\gamma_{ab} + J^a \delta A_a \right) , \tag{3.2}$$

where one can write

$$t^{ab} \doteq \frac{2}{\sqrt{-\gamma}} \frac{\delta S_g}{\delta \gamma_{ab}} , \tag{3.3}$$

$$J^a \doteq \frac{1}{\sqrt{-\gamma}} \frac{\delta S_g}{\delta A_a} . \tag{3.4}$$

It becomes evident from (3.2) that if one takes Σ to be the boundary, one recovers the usual definitions of the energy-momentum tensor and the gauge current according to the holographic dictionary.

The constraints are then found by considering the gauge and diffeomorphism variation in (3.2), and they read

$$D_a J^a = 0 ,$$
$$D_a t^{ab} - F^{ba} J_a = 0 .$$

We can see they are formally equal to the Ward identities for the $U(1)$ current and the stress tensor in standard field theory.

These membrane observables may be defined at every Σ, but they are in general non trivially related between one surface and the other. Finding such a relation would still involve the solution of the dynamical equations. In terms of these membrane currents, though, the equations of motion can be expressed as first order partial differential equations.

3.2.2 Wald construction and conservation laws

Relating the full membrane currents between different membranes requires solving the equations of motion, as we have just commented. We can however exploit some symmetries to connect them partially. In particular, there are conserved charges that can be used to relate the zero modes of these currents between different membranes. They can be derived through the Wald construction as follows.

We first consider the variation of the bulk action

$$\delta S_\Sigma = \int E \delta \Phi + d\theta \tag{3.5}$$

with respect to the set of fields Φ, which gives us the equations of motion E. The result is only defined up to a total derivative. The d-form θ on which it acts is called the presymplectic form.

Now suppose that the variation is made with respect to a diffemorphism generated by ξ and a gauge variation generated by α, such that $\delta = \delta_\xi + \delta_\alpha$. From now on we include subscripts ξ and α to denote which contributions arise for each transformation. The variation of any form with respect to diffeomorphisms is given by

$$\delta_\xi \omega = \mathcal{L}_\xi \omega = i_\xi d\omega + d\left(i_\xi \omega\right) , \tag{3.6}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the generator ξ. The second identity is known as Cartan formula and it is a quite remarkable result, because it relates interior product, exterior derivative and Lie derivative.

We now go back at (3.5) and assume the action is covariant, that it has no anomalies. Applying (3.6) to the Lagrangian L, the first term gives no contribution because L is a $d+1$ form and we can write

$$\delta_\xi L = d i_\xi L \, . \tag{3.7}$$

Using this, we can rewrite the variation of the action as an on-shell closure relation

$$dJ_{\xi,\alpha} \doteq 0$$

for the Noether current

$$J_{\xi,\alpha} \doteq \theta_{\xi,\alpha} - i_\xi L \, . \tag{3.8}$$

This current has various ambiguities, as pointed out by Wald [148]. In particular, we can add to it an exact form dk without spoiling its closedness

$$J_{\xi,\alpha} \to \hat{J}_{\xi,\alpha} = \theta_{\xi,\alpha} - i_\xi L + dk_{\xi,\alpha} \, .$$

We can fix this ambiguity by demanding that it is zero on-shell

$$\hat{J}_{\xi,\alpha} \doteq 0 \, ,$$

which can always be done because $\hat{J}$ is locally exact on-shell due to Poincaré's lemma. This vanishing of the Noether current implies also that k is closed on-shell when the gauge transformations or diffemorphisms leave the background solution invariant. This particular choice of k is called Komar form. We briefly review how it can be done in the following.

Let us first consider pure gauge transformations. The transformation of the gauge field is given by $\delta_\alpha A = d\alpha$, so a transformation that leaves the field invariant will be defined by a constant α. For such transformations the presymplectic form vanishes identically. This happens because all its contributions include one variation of the fields and it is therefore proportional to $d\alpha$. Then, plugging this in (3.8) we get our desired closure relation

$$0 \doteq \hat{J}_\alpha \doteq dk_\alpha \, , \qquad \text{for } \alpha = \text{const} \, .$$

The closed form k can be integrated on a $(d-1)$-dimensional hypersurface and one recovers the Gauss law. However, in holography we can make a different use of this quantity. If we integrate the Komar form on a d dimensional hypersurface Σ of constant $\hat{r}$, this integral gives the flux of k_α through Σ. Expressed in components, these fluxes read

$$I^a \doteq \int_\Sigma d^d x \sqrt{-g} \left(n_\mu k_\alpha^{\mu\nu} P_\nu^a\right) , \tag{3.9}$$

where $k_\alpha^{\mu\nu}$ is the Hodge dual of k_α, n^μ is a vector normal to Σ and P_μ^a is an orthogonal projector onto Σ. In the Fefferman-Graham gauge, they reduce to $n = \partial_{\hat{r}}$ and $P^a_\mu = \delta^a_\mu$. Due to the simplicity of this form, we will sometimes omit them for the sake of clarity.

The closure relation in terms of the Hodge dual of k_α reads $\partial_\mu k^{\mu\nu}_\alpha \doteq 0$. It allows us to find an expression for the radial derivative of the fluxes

$$\partial_{\hat{r}} I^a \doteq \int_\Sigma d^d x \partial_b \left(\sqrt{-\gamma} k^{ab}_\alpha \right) ,$$

where we have introduced the induced metric γ. In most cases, when the surface terms go to zero sufficiently fast, the right-hand side vanishes giving rise to a radial conservation equation for the fluxes. We will see below that the evaluation of that integral requires some extra care when some DC modes are present. In the context of holography this radial conservation can be interpreted as an RG equation for I^a. More on this will be commented below.

In the case of diffeomorphisms the natural generalization is to impose a Killing condition on the set of bulk fields $\mathcal{L}_\xi \Phi \doteq 0$, which assures invariance under diffeomorphisms. This is slightly different from the trivial gauge transformations given by constant α because it is not guaranteed that Killing equations are satisfied for generic backgrounds. Thus, the construction of the diffeomorphism charge imposes nontrivial restrictions on the possible solutions.

The presymplectic current is by construction proportional to the variation of the fields, so it vanishes on-shell if there exists a Killing vector. The extra term in (3.8) proportional to the Lagrangian can always be written as a total derivative as long as L is covariant. Because of Poincaré's lemma, (3.7) induces that locally

$$i_\xi L \doteq d\zeta_\xi .$$

Then, one can define a new quantity $k'_\xi = k_\xi - \zeta_\xi$ which is a closed $(d-1)$-form

$$0 \doteq \hat{J}_\xi \doteq dk'_\xi .$$

In a similar way to the gauge case, one can define a flux H^a by integrating k' on a d dimensional surface

$$H^a = \int_\Sigma d^d x \sqrt{-g} \left(n_\mu k'^{\mu\nu}_\xi P^a_\nu \right) . \tag{3.10}$$

Analogously to I^a, we can express the radial derivative of the flux as

$$\partial_{\hat{r}} H^a \doteq \int_\Sigma d^d x \partial_b \left(\sqrt{-\gamma} k'^{ab}_\xi \right) .$$

Due to our application to systems in equilibrium, we must take our Killing vector associated to time translations, i.e. $\xi = \partial_t$. Fortunately, this choice allows us to apply the construction to the heat current and, subsequently, to the energy current. Therefore, this result is usually presented as a spatial flux H^i. For these components the term proportional to the Lagrangian in the Noether current gives no contribution.

The holographic importance of this construction becomes apparent once the conserved fluxes are written as functions of our previously defined membrane currents.

We will now work out in detail the case of Einstein-Maxwell theory. This will allow us to better understand the workings of the construction for simpler theories before moving on to cases with anomalies.

3.2.3 Preliminary example: Einstein-Maxwell theory

Let us consider Einstein-Maxwell theory with cosmological constant in $d+1$ spacetime dimensions. The action is given by

$$S_g = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\left(R - \frac{2\Lambda}{L^2}\right) - \frac{1}{4}\int d^{d+1}x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + S_{GH}\,,$$

where

$$S_{GH} = \frac{1}{8\pi G}\int_\Sigma d^dx\sqrt{-\gamma}K\,, \tag{3.11}$$

is the Gibbons-Hawking counterterm, which assures a well defined variational problem at a general Cauchy surface. The coupling G is Newton's constant and L represents the AdS length.

Using (3.3) and (3.4), we can obtain

$$t^{ab} = -\frac{1}{8\pi G}\left(K^{ab} - \gamma^{ab}K\right)\,, \tag{3.12}$$

$$J^a = 2n_\mu \frac{\partial L}{\partial F_{\mu\nu}}P^a_\nu = -n_\mu F^{\mu\nu}P^a_\nu\,. \tag{3.13}$$

which are the Brown-York tensor and the membrane current, respectively.

We can also recognize the presymplectic form from (3.5). For this theory and, in general, for any action that depends on the gauge field only through the curvature, the presymplectic form associated to gauge transformations is given by

$$\theta_\alpha = \frac{\partial L}{\partial F}d\alpha\,,$$

whose Hodge dual reads

$$\theta^\mu_\alpha = 2\frac{\partial L}{\partial F_{\mu\nu}}\partial_\nu\alpha\,.$$

The equations of motion read $\nabla_\mu \frac{\partial L}{\partial F_{\mu\nu}} = 0$. Therefore, we can easily exploit the ambiguity in the definition of the Noether current in order to make it vanish on-shell. If we add $\nabla_\nu k^{\mu\nu}_\alpha$ to it, for

$$k^{\mu\nu}_\alpha = -2\frac{\partial L}{\partial F_{\mu\nu}}\alpha\,,$$

the Noether current finally gives

$$\hat{J}^\mu_\alpha = -2\alpha\nabla_\nu\frac{\partial L}{\partial F_{\mu\nu}} \doteq 0\,.$$

Let us now make the connection between the fluxes (3.9) for $k_\alpha^{\mu\nu}$ and the current (3.13). One can immediately see that both results match for $\alpha = 1$. Therefore, the radially conserved flux I^a is, as noted by [44], just the flux of such currents

$$I^a = -\int_\Sigma d^d x\sqrt{-\gamma}J^a\,.$$

By construction, it coincides with the flux of the dual CFT's $U(1)$ current when Σ is equal to the boundary. The fact that it does not change along the holographic direction means that we are able to obtain the dual theory's result completely in terms of the horizon data

Diffeomorphisms require a more involved computation. For theories depending only on the curvatures $R^\alpha_{\beta\mu\nu}$ and $F_{\mu\nu}$ the presymplectic current is

$$\theta^\mu_\xi = 2\frac{\partial L}{\partial R_{\mu\nu}{}^\rho{}_\sigma}\delta\Gamma_{\nu\sigma}^\rho + 2\frac{\partial L}{\partial F_{\mu\nu}}\delta A_\nu\,.$$

For the Einstein-Maxwell theory, in particular, θ is given by [83]

$$\theta^\mu_\xi = \frac{1}{16\pi G}g^{\mu\nu}g^{\rho\sigma}\left(\nabla_\rho\delta g_{\nu\sigma} - \nabla_\nu\delta g_{\rho\sigma}\right) + 2\frac{\partial L}{\partial F_{\mu\nu}}\delta A_\nu\,.$$

Particularizing for diffeomorphisms, it becomes

$$\theta^\mu_\xi = \frac{1}{16\pi G}\nabla_\nu\left(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu\right) + 2\frac{\partial L}{\partial F_{\mu\nu}}\left(\nabla_\nu(A_\rho\xi^\rho) + F_{\nu\rho}\xi^\rho\right)\,. \tag{3.14}$$

The Noether current can be constructed by adding $\xi^\mu L$ to this term. Then, to find the Komar form we look for a total derivative that added to the Noether current makes it vanish on-shell. A careful rewriting shows that the derivative of

$$k^{\mu\nu}_\xi = \frac{1}{4\pi G}\nabla^{[\mu}\xi^{\nu]} + A_\alpha\xi^\alpha\frac{\partial L}{\partial F_{\mu\nu}} \tag{3.15}$$

reduces the Noether current to a linear combination of Einstein and Maxwell equations and one arrives to the spatial flux density

$$2n_\mu k^{\mu\nu}_\xi P^i_\nu = -\frac{1}{8\pi G}K^{ib}\xi_b + A_c\xi^c J^i\,,$$

where we have used that in Fefferman-Graham coordinates $\nabla_{\hat{r}}\xi^a = -K^{ab}\xi_b$, if ξ^a is radially independent.

At this point we are ready to compute the conserved flux associated to diffeomorphisms (3.10). It coincides with one half of the flux of the membrane heat current

$$H^i = \frac{1}{2}\int_\Sigma d^d x\sqrt{-\gamma}\left(t^i{}_b\xi^b + A_c\xi^c J^i\right) = \frac{1}{2}\int_\Sigma d^d x\sqrt{-\gamma}Q^i\,. \tag{3.16}$$

On the conformal boundary, this result gives the dual theory's heat current flux. The factor of $1/2$ could be absorbed in a redefinition of ξ.

In this discussion we have in principle considered the Einstein-Maxwell Lagrangian. However, in our computations of the membrane currents and the presymplectic currents we have not assumed any particular dependence on the field-strength $F_{\mu\nu}$. On the contrary, we have assumed the gravitational dependence to be that of the Einstein-Hilbert action. Therefore, strictly speaking, all the expressions would remain valid if we performed the construction for other Lagrangians which are functions of the curvature $F_{\mu\nu}$ and possibly to uncharged matter too.

The construction of the Komar charges would remain unchanged under the addition of counterterms to the action like the ones used for holographic renormalization. This is a consequence of the on-shell nature of the construction. Furthermore, this assures that, as long as the charges are finite on the horizon, their associated one-point functions need no renormalization.

Another way of understanding it is that one can usually rearrange the equations of motion in order to find divergence free currents that are equivalent to $k^{\mu\nu}$, like it was done for gauge transformations in the seminal work [82]. For diffeomorphisms, finding these divergence free currents is usually more cumbersome. It always requires considering Einstein's equations dotted with the Killing field ξ^{μ} and using various identities related to Lie derivatives, as in [43].

The construction for anomalous theories is slightly different. We now move on to cover how it can be extended to such cases.

3.3 Extension to anomalous theories

In holography anomalies are introduced in a very natural way by adding to the action Chern-Simons terms, as discussed in Section 2.4.2. Extra care is needed, though, when dealing with gravitational contributions to the anomaly [128]. When the constraints of the gravitational side are ADM decomposed, it can be seen that they do not match field theory's Ward identities due to some extra terms containing the extrinsic curvature. Fortunately, these terms do not survive once AdS asymptotics are imposed.

However, there is no reason to suppress these extrinsic contributions away from the conformal boundary. Furthermore, in this work we propose a way to interpret the resulting Ward identities. Our approach is looking for a redefinition of the membrane currents that allows us to obtain constraints for every constant radial coordinate hypersurface that are formally equal to the usual field theory Ward identities. We exploit the freedom to choose the definition of the current operators in the presence of anomalies and look for new currents that possess nice properties from the point of view of the RG. The way to choose a particular definition is by adding certain local counterterms in the boundary, but this does not affect the form of the Komar charges. This suggest that there exists a certain "preferred choice" in the definition of the currents from the point of view of the holographic RG.

Besides that, a theory that contains gravitational Chern-Simons terms does not have in general a well defined variational problem due to the presence of higher derivatives. This is of course extremely problematic for the construction of the space of solutions. This point is usually overlooked in holography because the gravitational anomaly coefficient λ is subleading in the large N expansion and therefore the variational problem may get corrected by other subleading terms. As far as we are concerned, understanding the stability of such solutions requires some more work. Related issues have been studied in AdS/CFT in the context of topologically massive gravity (TMG) [127]. In particular, [113] showed the presence of finite momentum instabilities for charged black-holes in five dimensions. In Appendix C of [36] we comment on how these works on TMG can be linked to the modifications of the membrane currents, but we will not include such discussion in this document because it is a detour from our main purpose. Since imposing asymptotic AdS boundary conditions from the beginning guarantees the stability of the solutions, we will take such boundary conditions and not make further efforts to study the definition of the Cauchy problem .

In the following we build on the construction from the previous section in order to find the different currents and their relation to the Komar charges. We will see that it is precisely the extrinsic terms that carry the relevant information regarding anomalous transport. We mainly focus on four dimensional field theories but, since the equations are known to follow a general structure, we will keep a general notation, as in [12].

3.3.1 Equations of motion and presymplectic current

Let us consider a $(d+1)$-dimensional gravitational theory of the form

$$\begin{aligned} S =& \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\left(R-\frac{2\Lambda}{L^2}\right)+\int d^{d+1}x\sqrt{-g}L_{mat} \\ &+\int d^{d+1}x\sqrt{-g}I_{CS}\left[A,F,\mathbf{\Gamma},\mathbf{R}\right]+S_{GH}\,, \end{aligned} \tag{3.17}$$

where L_{mat} denotes the matter Lagrangian and I_{CS} is the bulk Chern-Simons action as defined in (2.22). We have also included S_{GH}, which is the Gibbons-Hawking term given by (3.11) and is included to make the variational problem well-defined for the Einstein-Hilbert action. The matter Lagrangian is taken again to be a function of $F_{\mu\nu}$ only, although uncharged scalars could be added too. The presence of charged matter, on the contrary, would spoil the simple structure we will encounter below for the long wavelength limit of the theory, as will become evident in Chapter 5.

Let us now take the variation of the action to obtain the equations of motion and the presymplectic current

$$\delta I_{CS}=\left(\frac{\partial I_{CS}}{\partial F}\delta F+\frac{\partial I_{CS}}{\partial A}\delta A\right)+\left(\frac{\partial I_{CS}}{\partial d\mathbf{\Gamma}}d\delta\mathbf{\Gamma}+\frac{\partial I_{CS}}{\partial \mathbf{\Gamma}}\delta\mathbf{\Gamma}\right)\,.$$

Integrating by parts leads to

$$\delta I_{CS}=\Sigma^{\mu}\delta A_{\mu}+E^{\mu\nu\rho}\delta\Gamma^{\rho}_{\mu\nu}+d\left(\frac{\partial I_{CS}}{\partial F}\delta A+\frac{\partial I_{CS}}{\partial d\mathbf{\Gamma}}\delta\mathbf{\Gamma}\right)\,,$$

where

$$
\begin{aligned}
\Sigma^{\mu} =& \frac{\partial I_{CS}}{\partial A_{\mu}} - 2\nabla_{\nu}\frac{\partial I_{CS}}{\partial F_{\nu\mu}}\,, \\
E^{\mu\nu}{}_{\rho} =& \frac{\partial I_{CS}}{\partial \Gamma^{\rho}_{\mu\nu}} - 2\nabla_{\sigma}\frac{\partial I_{CS}}{\partial R_{\sigma\mu}{}^{\rho}{}_{\nu}}\,,
\end{aligned}
$$

and Σ^{μ} is known as the Hall current. The usual form of Einstein's equations is obtained as the variation with respect to the metric and the Chern-Simons action in (3.17) is a function of connections and curvatures, as opposed to $I_{CS}[A, dA, \mathbf{\Gamma}, d\mathbf{\Gamma}]$. Therefore, we need to rearrange the second and fourth terms such that the equations of motion appear contracted with variations of the metric and the derivatives are taken with respect to $\mathbf{R}$ and not $d\mathbf{\Gamma}$. The final result for the variation is

$$
\delta I_{CS} = \Sigma^{\mu}\delta A_{\mu} - \nabla_{\rho}\Sigma^{\mu\nu\rho}\delta g_{\mu\nu} + \nabla_{\mu}\left(2\frac{\partial I_{CS}}{\partial F_{\mu\nu}}\delta A_{\nu} + \frac{\partial I_{CS}}{\partial \partial_{\mu}\mathbf{\Gamma}_{\rho}}\delta\mathbf{\Gamma}_{\rho} + \Sigma^{\alpha\beta\mu}\delta g_{\alpha\beta}\right)\,,
$$

where

$$
\Sigma^{\mu\nu\rho} = \frac{1}{2}\left(E^{\mu\nu\rho} + E^{\mu\rho\nu} - E^{\rho\mu\nu}\right)
$$

is the spin current. From the full variation of the action we can obtain the equations of motion

$$
2\nabla_{\mu}\frac{\partial L_{mat}}{\partial F_{\mu\nu}} = \Sigma^{\nu}\,, \tag{3.18}
$$

$$
G^{\mu\nu} - \Lambda g^{\mu\nu} = T^{\mu\nu}_{mat} + \nabla_{\rho}\Sigma^{(\mu\nu)\rho}\,, \tag{3.19}
$$

where $T^{\mu\nu}$ is the matter stress tensor. We have also found the Chern-Simons part of the presymplectic current

$$
\theta^{\mu}_{CS} = 2\frac{\partial I_{CS}}{\partial F_{\mu\nu}}\delta A_{\nu} + 2\frac{\partial I_{CS}}{\partial \mathbf{R}_{\mu\rho}}\delta\mathbf{\Gamma}_{\rho} + \Sigma^{\alpha\beta\mu}\delta g_{\alpha\beta}\,. \tag{3.20}
$$

In the five dimensional theory (2.22), the Hall and spin currents read

$$
\begin{aligned}
\Sigma^{\mu} &= \kappa\epsilon^{\mu\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\,, \\
\Sigma^{\mu\nu\rho} &= 2\lambda\epsilon^{\mu\alpha\beta\gamma\delta}F_{\alpha\beta}R^{\nu\rho}{}_{\gamma\delta}\,.
\end{aligned}
$$

In general both currents are covariant expressions even if they come from a non-covariant action. This intriguing feature motivated a series of works on the extension of the Wald construction to theories with a noncovariant Lagrangian [138, 29, 12]. Their construction includes a large part of the technical tools we use in this work and it was key to us in order to extend our results past perturbation theory. We review it below and include the explicit derivative of the Komar charges in the relevant cases.

3.3.2 Anomalous membrane currents

The explicit form of the constraint equations can be found through variation of the on-shell action. In theories supplemented with gravitational Chern-Simons terms there is in general no counterterm à la Gibbons-Hawing that makes the variational problem well defined. There are instabilities of the Ostrogradski type because of the appearance of higher derivatives, but it is even more worrisome the fact that an ill-defined variational problem can spoil the unitarity of the dual theories. All the effects it might have are still not clear, but some cases have already been studied, like the one in AdS_3 [127].

The on-shell variation takes the general form

$$\delta S_\Sigma = \int_\Sigma d^d x \sqrt{-\gamma} \left(\frac{1}{2} t^{ab} \delta\gamma_{ab} + \frac{1}{2} u^{ab} \delta K_{ab} + l^{ab}{}_c \delta\hat{\Gamma}^c_{ab} + \mathcal{J}^a \delta A_a \right) , \tag{3.21}$$

where the precise form of the various quantities depends on the choice of Chern-Simons action. We separate the variation with respect to the connection from the one with respect to the induced metric because, while t^{ab} only contains gauge and diffeomorphism invariant quantities, $l^{ab}{}_c$ might not be covariant if anomalies are present.

The presence of the connections A and $\mathbf{\Gamma}$ in the Chern-Simons action spoil the gauge and diffeomorphism invariance of the theory. This leads to an anomalous variation of the action

$$\bar{\delta} S_\Sigma \doteq \int_\Sigma d^d x \sqrt{-\gamma} n^\mu \left(\alpha \frac{\partial I_{CS}}{\partial A_\mu} + \Lambda \frac{\partial I_{CS}}{\partial \mathbf{\Gamma}_\mu} \right) , \tag{3.22}$$

where the variation $\bar{\delta}$, which includes simultaneously gauge transformations and diffeomorphisms, is defined as

$$\begin{aligned} \bar{\delta} A &= d\alpha \, , \\ \bar{\delta} \mathbf{\Gamma} &= d\mathbf{\Lambda} + [\mathbf{\Lambda}, \mathbf{\Gamma}] \, , \end{aligned}$$

such that $(\Lambda)^\mu_\nu = \partial_\nu \xi^\mu$ is the connection's gauge parameter. The anomalous variation (3.22) must be equal to (3.21) when the variations of the fields are particularized to gauge transformations and diffeomorphisms.

Let us now treat the gauge and diffeomorphism cases separately.

Anomalous gauge constraint

The consistent current $\mathcal{J}^a$ is given by

$$\mathcal{J}^a \doteq \frac{1}{\sqrt{-\gamma}} \frac{\delta S_g}{\delta A_a} = 2 n_\mu \left(\frac{\partial L_{\mathrm{mat}}}{\partial F_{\mu\nu}} + \frac{\partial I_{CS}}{\partial F_{\mu\nu}} \right) P^a_\nu \, . \tag{3.23}$$

Integrating by parts the gauge variation of (3.21) and equating it to (3.22), we get the anomalous constraint for the consistent current

$$D_a \mathcal{J}^a = -n_\mu \frac{\partial I_{CS}}{\partial A_\mu} \, . \tag{3.24}$$

This constraint has a slightly different interpretation depending on whether a possible gravitational anomaly lies in the diffeomorphism sector, like in (2.24), or in the gauge sector, like in (2.22). For pedagogical reasons, we treat both cases separately in the following.

Let us consider the first case, in which the mixed anomaly lies in the diffeomorphism sector. No gravitational characteristic class appears on the right hand side of (3.24), but the definition of the consistent membrane current picks up some contribution from the Chern Simons term. In particular, the constraint now reads

$$D_a \mathcal{J}^a = -\frac{\kappa}{3}\epsilon^{abcd} F_{ab} F_{cd} \,,$$

and we can decompose the consistent current in three different parts

$$\mathcal{J}^a = 2 n_\mu \frac{\partial L_{\text{mat}}}{\partial F_{\mu\nu}} P^a_\nu + J^a_{CSK} + J^a_{BZ} \,.$$

The first one is the contribution that we already obtained for the Einstein-Maxwell theory and comes from the first term in (3.23). It is present in all the possible definitions of the current and it has no ambiguity associated because it is not produced by the anomaly. The other two parts come from the ADM decomposition of the Chern-Simons contribution to (3.23). Contributions involving extrinsic curvatures are collected in the extrinsic Chern-Simons current J^a_{CSK}, and J^a_{BZ} is the usual Bardeen-Zumino polynomial needed to define covariant currents, which we can recognize as the purely intrinsic part.

For the theory with mixed anomaly introduced above, the Bardeen-Zumino polynomial receives two contributions from the gauge and gravitational sectors, such that $J^a_{BZ} = J^a_A + J^a_{\hat{\Gamma}}$. The different parts of the current are given by

$$\begin{aligned} J^a_{CSK} &= -8\lambda\epsilon^{abcd} K^f_b D_c K^f_d \,, \\ J^a_A &= \frac{4}{3}\kappa\epsilon^{abcd} A_b F_{cd} \,, \\ J^a_{\hat{\Gamma}} &= 4\lambda\epsilon^{abcd} \mathrm{Tr}\left(\hat{\Gamma}_b \partial_c \hat{\Gamma}_d + \frac{2}{3}\hat{\Gamma}_b \hat{\Gamma}_c \hat{\Gamma}_d\right) \,. \end{aligned}$$

Since J^a_A is a well-known result, let us now make some further comments only on the relevant aspects of the different gravitational contributions.

Among the parts coming from the gravitational anomaly, the extrinsic current is the only one which is first order in spacetime derivatives. That makes it a good candidate to encode the anomalous horizon fluctuations involving the vorticity. However, a simple asymptotic analysis shows that J^a_{CSK} vanishes identically at the conformal boundary [12]. Thus, any relevant effect must be dynamically generated inside the bulk. Furthermore, the extrinsic current is perfectly diffeomorphism and gauge invariant from the point of view of the membrane Σ, so it cannot contribute in any way to the anomaly. Nothing prevents it from having physical consequences, though.

The intrinsic contribution $J^a_{\hat{\Gamma}}$ can be seen to give rise to the mixed anomaly, according to

$$D_a J^a_{\hat{\Gamma}} = \lambda \epsilon^{abcd} \mathrm{Tr}\left(\hat{\mathbf{R}}_{ab}\hat{\mathbf{R}}_{cd}\right) .$$

Therefore, the mixed anomaly can be moved to the gauge sector by a Bardeen counterterm of the form

$$B_\Sigma = -\int_\Sigma d^d x \sqrt{-\gamma}\, A_a J^a_{\hat{\Gamma}} = -\int_\Sigma d^d x \sqrt{-\gamma}\, 4\lambda \epsilon^{abcd} A_a \mathrm{Tr}\left(\hat{\mathbf{\Gamma}}_b \partial_c \hat{\mathbf{\Gamma}}_d + \frac{2}{3}\hat{\mathbf{\Gamma}}_b \hat{\mathbf{\Gamma}}_c \hat{\mathbf{\Gamma}}_d\right) . \tag{3.25}$$

In that case the consistent current loses its $J^a_{\hat{\Gamma}}$ contribution, while the extrinsic part remains as a defining feature of the membrane current. We could therefore write the resulting constraint equation as

$$D_a\left(\mathcal{J}^a - J^a_{\hat{\Gamma}}\right) = -\frac{\kappa}{3}\epsilon^{abcd} F_{ab} F_{cd} - \lambda \epsilon^{abcd} \mathrm{Tr}\left(\hat{\mathbf{R}}_{ab}\hat{\mathbf{R}}_{cd}\right) , \tag{3.26}$$

which is formally equivalent to the usual consistent form of the $U(1)^3$ anomaly plus the mixed gauge-gravitational anomaly.

Let us now consider the other case, which is also the usual approach to this theory in the literature. We consider from the beginning a Chern-Simons action (2.22) that is diffeomorphism invariant, like in [108]. The right-hand side of the constraint equation now obtains a contribution from the gravitational sector that reads

$$n_\mu \frac{\partial I_{CS}}{\partial A_\mu}\bigg|_{grav} = -\lambda \epsilon^{abcd} \mathrm{Tr}\left(\hat{\mathbf{R}}_{ab}\hat{\mathbf{R}}_{cd}\right) - D_a J^a_{CSK} .$$

and one has to add a counterterm explicitly dependent on the extrinsic data

$$S_{CSK} = \int_\Sigma d^d x \sqrt{-\gamma}\, A_a J^a_{CSK} = -\int_\Sigma d^d x \sqrt{-\gamma}\, 8\lambda \epsilon^{abcd} A_a K^f_b D_c K^f_d , \tag{3.27}$$

in order to recover the correct Ward identity and obtain the correct definition of the membrane currents.

Without the inclusion of such a counterterm the horizon physics are the same. However, one might feel tempted to explain the anomalous fluctuations as a response to the "thermal" anomaly

$$\mathcal{A}_{thermal} \equiv 64\pi^2 T^2 E_g \cdot B_g ,$$

where E_g and B_g are the gravitoelectric and gravitomagnetic fields on the horizon, as for example in [33]. We prefer not to think about it this way, since this is not a true anomaly, as it comes from the divergence of a physical current and it is state dependent.

At this point we can go back to the different possible definitions of the anomalus currents discussed in Section 2.1.6. We can define for each membrane a consistent current $\mathcal{J}^a$ that is obtained as the on-shell variation of the action

$$\mathcal{J}^a \doteq \frac{1}{\sqrt{-\gamma}} \frac{\delta S_g}{\delta A_a},$$

a covariant current J^a whose variation under gauge transformations and diffeomorphisms is zero

$$\delta_{\alpha,\xi} J^a = 0,$$

and even a conserved current J^a_{cons} whose divergence is zero

$$D_a J^a_{cons} = 0.$$

The explicit expressions, considering the possibility of adding a Bardeen counterterm cB_Σ to the action, read

$$\mathcal{J}^a = 2n_\mu \frac{\partial L_{\text{mat}}}{\partial F_{\mu\nu}} + J^a_{CSK} + J^a_A + (1-c) J^a_{\hat{\Gamma}}, \tag{3.28}$$

$$J^a = 2n_\mu \frac{\partial L_{\text{mat}}}{\partial F_{\mu\nu}} P^a_\nu + J^a_{CSK}, \tag{3.29}$$

$$J^a_{cons} = J^a + J^a_{\hat{\Gamma}} + \frac{3}{2} J^a_A. \tag{3.30}$$

The factor $3/2$ in the definition of the conserved current depends on the dimensionality of the theory. In general it can be extracted by writing the covariant $U(1)^3$ anomaly as a total divergence.

Anomalous diffeomorphism constraint

The diffeomorphism constraints allow us to define a consistent membrane stress tensor. We might encounter problems in the construction due to the presence of higher derivative terms. These terms vanish at the boundary due to the asymptotic AdS boundary conditions but they could in principle have a contribution inside the bulk. We will treat the metric and the extrinsic curvature, which is essentially given by a radial derivative of the metric, as independent fields and this will help us achieve our goal of defining valid membrane currents.

Since the variation under diffeomorphisms of the different fields is given by the Lie derivative with respect to the diffeomorphism generator ξ, we can use the following formulas

$$\begin{aligned}
\mathcal{L}_\xi \gamma_{ab} &= D_a \xi_b + D_b \xi_a, \\
\mathcal{L}_\xi A_a &= \partial_a \left(A_b \xi^b \right) + F_{ab} \xi^b, \\
\mathcal{L}_\xi K_{ab} &= \xi^c D_c K_{ab} + D_a \xi^c K_{cb} + D_b \xi^c K_{ac}.
\end{aligned}$$

This allows us to extract the diffeomorphism constraints as the terms proportional to the Killing field ξ^a in the on-shell variation of the action (3.21) under diffeomorphisms

$$\delta_\xi S_\Sigma = \int_\Sigma d^d x \sqrt{-\gamma} \left(D_a \Theta^{ab} - F^{bc} \mathcal{J}_c + A^b D_c \mathcal{J}^c + \Delta^b \right) \xi_b, \tag{3.31}$$

where we have defined for convenience

$$\Theta^{ab} = t^{ab} + u^{ac}K_c^b + D_d\left(l^{d(ba)} + l^{(adb)} - l^{(ab)d}\right),$$
$$\Delta^b = -\frac{1}{2}u^{ac}D^b K_{ac},$$

and $\mathcal{J}^a$ is given in (3.28). Furthermore, if we divide the stress tensor into the Einstein-Hilbert and anomalous parts, as in $t^{ab} = t^{ab}_{EH} + t^{ab}_{\lambda}$, we get the following expressions

$$t^{ab}_{\lambda} = n_\mu\left(\Sigma^{ab\mu} + \Sigma^{ba\mu}\right), \tag{3.32}$$
$$u^{ab} = 2n^\mu n^\nu\left(\frac{\partial I_{CS}}{\partial R_{\mu a\nu}{}^{b}} - \frac{\partial I_{CS}}{\partial R_{\mu a}{}^{b}{}_{\nu}}\right),$$
$$l^{ab}{}_{c} = 2n^\nu\frac{\partial I_{CS}}{\partial R_{\nu a}{}^{c}{}_{b}},$$

which follow from the ADM decomposition of the radial component of the presymplectic current (3.20). The remaining part of the stress tensor t^{ab}_{EH} is given by (3.12).

Let us now assume a counterterm cB_Σ, where B_Σ is given by (3.25), has been added to the action in order to move the anomaly to the gauge sector. Apart from the consistent current, as above, this only affects Θ^{ab} in the following way

$$\Theta^{ab} \to \Theta^{ab} - cD_d\left(l^{d(ba)} + l^{(adb)} - l^{(ab)d}\right) - 4c\lambda\epsilon^{mnp(a}D_e\left(A_m R_{np}{}^{b)e}\right).$$

The constraint equations are then derived by equating $\delta_\xi S_\Sigma$ in (3.31) to the consistent diffeomorphism anomaly $\mathcal{A}^b$, which is obtained from the second term in (3.22) when it is also rearranged as a term proportional to ξ_b plus total derivatives. The resulting constraints take the general form

$$C^b = D_a\Theta^{ab} - F^{bc}\mathcal{J}_c + A^b D_c\mathcal{J}^c + \Delta^b - (1-c)\mathcal{A}^b \doteq 0. \tag{3.33}$$

This form of the consistent constraint suggests that Θ^{ab} must be interpreted as the consistent membrane stress tensor. For the particular theory we have chosen, the anomaly is

$$\mathcal{A}^b = 2\lambda\gamma^{bc}\frac{1}{\sqrt{-\gamma}}\partial_a\left(\sqrt{-\gamma}\epsilon^{defg}F_{de}\partial_f\Gamma^a_{gc}\right).$$

We can also exploit the freedom to define different operators in anomalous theories for the stress tensor. The covariant stress tensor T^{ab}, for example, can be found reexpressing the Ward identity in terms of covariant quantities. In order to do this, we need to drop from the stress tensor the explicit dependence on the gauge field A_m and also the dependence on the noncovariant variation with respect to Christoffel's connection $l^{ab}{}_{c}$, as in

$$T^{ab} = \Theta^{ab} - (1-c)D_d\left(l^{d(ba)} + l^{(adb)} - l^{(ab)d}\right) + 4c\lambda\epsilon^{mnp(a}D_e\left(A_m R_{np}{}^{b)e}\right).$$

The final expression for the covariant membrane stress tensor is simply

$$T^{ab} = t^{ab} + u^{ac} K_c^b \,. \tag{3.34}$$

and its associated Ward identity, written also in terms of the covariant current (3.29), reads

$$D_a T^{ab} - F^{bc} J_c + \Delta^b + 2\lambda \epsilon^{cdef} D_a \left(F_{cd} R^{ab}{}_{ef} \right) = 0 \,.$$

The last term of this equation, which is the result of combining the consistent anomaly with the pieces that distinguish consistent operators from covariant operators, must be interpreted as the covariant diffeomorphism anomaly. We can introduce inside the definition of the membrane stress tensor a term that cancels the last total derivative in the constraint equation. We propose this must be the conserved stress tensor T^{ab}_{cons} analogous to the conserved current, which reads

$$T^{ab}_{cons} = T_{ab} + 2\lambda \epsilon^{cdef} F_{cd} R^{ab}{}_{ef} \,.$$

As the energy-momentum tensor now contains an antisymmetric part, we have ended up with a Lorentz anomaly for this definition of the membrane stress tensor.

From the intuition gained in the preliminary example from Section 3.2.3, we know that the quantity that appears in the conserved flux associated to diffeomorphisms is the heat current. Therefore, it will prove more convenient for our purposes to work with the heat current which we define as

$$Q^a = \Theta^a{}_b \xi^b + A_b \xi^b \mathcal{J}^a \,. \tag{3.35}$$

Its conservation is only spoiled by the diffeomorphism anomaly

$$D_a Q^a = (1-c) \mathcal{A}_b \xi^b \,. \tag{3.36}$$

The right-hand side can always be cancelled by taking $c = 1$, which is the choice of counterterms that moves the anomaly completely to the gauge sector.

We have proposed a new definition for the heat current (3.35) as the result of exchanging $t^a{}_b$ by $\Theta^a{}_b$ in the usual definition, given for example in (3.16). However, this new definition produces a constraint, given by (3.36), that is formally equal to the usual field theory's constraint for the heat current. We thus understand this as a consistency check for our proposal of Θ_{ab} as the membrane consistent stress tensor.

Analogously to the other membrane currents, we can define a conserved heat current Q^a_{cons} which is divergence-free. We will skip the construction here but it can be found in Appendix A of [36]. The final result reads

$$Q^a_{cons} = \Theta^a{}_b \xi^b + A_c \xi^c \mathcal{J}^a - (1-c) \frac{1}{\sqrt{-\gamma}} \partial_b \left(\sqrt{-\gamma} l^{ba}{}_c \xi^c \right) + 2(1-c) l^{(ab)}{}_c \Lambda^c_b \,.$$

Now that all the different relevant quantities have been derived, let us introduce the explicit expressions for the Chern-Simons action given by (2.24). The different components of the membrane stress tensor read

$$
\begin{aligned}
t^{ab}_{EH} &= -\frac{1}{8\pi G}(K^{ab} - K\gamma^{ab})\,, \\
t^{ab}_{\lambda} &= -8\lambda\epsilon^{mnp(a}\left(2D_n K^{b)}_p F_{\hat{r}m} + \gamma^{b)l}\dot{K}_{ln}F_{pm} - F_{pm}K^{b)}_l K^l_n\right)\,, \\
u^{ab} &= 8\lambda\epsilon^{mnp(a}F_{mn}K^{b)}_p\,, \\
l^{ab}{}_c &= 2\lambda\epsilon^{amnp}F_{mn}\Gamma^b_{pc}\,.
\end{aligned}
$$

Before moving on to the construction of the Komar charges, let us make some remarks on the form of the constraint equations (3.33). First of all, it is important to notice that our candidate Θ^{ab} for the consistent membrane stress tensor does not coincide with the well-known Brown-York prescription

$$
\Theta^{ab} \neq \frac{2}{\sqrt{-\gamma}}\frac{\delta S_g}{\delta\gamma_{ab}}\,,
$$

precisely due to the additional term coming from the extrinsic curvature contributions. One may wonder if this is an artifact of the way in which we have organized the various fields in the constraint equation, since one could in principle include the modification into a redefinition of Δ^b. However, as we have already discussed, that term is necessary in the definition of the heat current (3.35) to obtain the right constraint.

Another important consistency check for our proposal is that it gives the right near boundary limit for the dual stress tensor $\langle T^a{}_b\rangle$, which is easily computed from (3.21) using that asymptotically $\delta K^a{}_b = \delta\gamma^a{}_b$ plus subleading terms. No other contributions arise as long as strictly AdS boundary conditions are imposed

$$
\langle T^a{}_b\rangle = \lim_{r\to\infty}\sqrt{-\gamma}\,(t^a{}_b + u^a{}_b) = \lim_{r\to\infty}\sqrt{-\gamma}\left(t^{ab} + u^{ac}K_{cb}\right) = \lim_{r\to\infty}\sqrt{-\gamma}\Theta^a{}_b\,.
$$

Furthermore, holographic renormalization would give no contribution to the new terms in the definition of the membrane stress tensor.

Yet another check can be performed by moving the anomalies entirely to the gauge sector and using the asymptotic expansion to compare to the known results for the bare stress tensor. The result from [117] is recovered.

Our prescription has been already applied successfully to systems where momentum relaxation is introduced in the presence of a mixed gauge gravitational anomaly. In this case, the higher derivative corrections stemming from u^{ab} are crucial for the restoration of the symmetry of the mixed two point functions in the presence of a magnetic field. This will be covered in Chapter 4.

However, the most interesting aspect of it is that the constraint equations (3.33) are expected to hold all along the bulk. In general, they just differ from the usual Ward identities due to the contribution Δ^b, which is only guaranteed to vanish in the conformal boundary. Such difference takes a suggestive form if we think of u^{ab} as the expectation value of an independent operator, associated with the mode excited by the extrinsic curvature. From our point of view this is reflected in the constraint equation, where

$$\Delta^b \sim u^{ac} D^b K_{ac}$$

appears in the form [operator times ∂(source)], also presented for example by $J^a F^b{}_a$. This structure suggests that this second operator, even if turned off at the conformal boundary, dynamically gets an expectation value as we slide through the bulk. This is somehow similar to the reasoning in [127], where such operator was indeed shown to survive at the conformal boundary for general solutions and lead to logarithmic correlators with the canonical stress tensor in AdS_3.

In the following section we match the membrane currents defined here to the conserved fluxes related to diffeomorphisms and gauge transformations.

3.3.3 Anomalous Wald construction and conservation laws

The construction of conserved fluxes associated to diffeomorphisms and gauge transformations in Chern-Simons theories is mainly taken from [12, 29] where the subtleties of the construction are also discussed at length. It is important to keep in mind that all the construction can be derived by explicitly using the equations of motion (3.18) and (3.19) and assuming the background solution to posses a Killing vector field. This is important because of two reasons that have already been pointed out. First, it makes the construction independent of the choice of counterterms used to define the membrane currents, and, second, the charges are consequently independent of any holographic renormalization procedure to remove divergences.

In the presence of bulk Chern-Simons terms the Wald construction is known to be plagued by ambiguities [29]. These are due to the lack of covariance for the bulk action, which introduces various subtleties in the extraction of the Komar form $k^{\mu\nu}$ from the Noether charge. In particular, the Chern-Simons terms give further contributions to the on-shell vanishing Noether current $J_{\xi,\alpha}$ which are proportional to the gauge parameters α and Λ. One can however still derive the conservation equation for the appropriate fluxes once a particular gauge is chosen. The charges thus derived are not covariant, but a covariant prescription for the differential Noether charge was given in [12], which allows one to unambiguously define the Wald entropy at a bifurcation surface. We will follow instead the ideas of [29] and the remarks in (5.3) of [12] which are closer in notation to the construction of Section 3.2.2. The price to pay will be a noncovariant expression for $k^{\mu\nu}$, which from our perspective is a feature rather than a bug. In fact, it allows us to link the flux conservation to the RG properties of the conserved current and the conserved stress tensor.

We are using the construction introduced in section 3.2.2 as a base, so we will only point out the differences. Again, the variation of the Lagrangian may be written as

$$\delta \left(L + I_{CS} \right) = E\delta\Phi + d\theta \, .$$

However, when the variation is taken with respect to diffeomorphisms (ξ) or gauge transformations (α), the Lagrangian does not change simply as a Lie derivative due to the lack of covariance of the Chern-Simons terms. Indeed a further piece $\Xi_{\xi,\alpha}$

arises because of the inflow of the consistent anomalies

$$\delta_{\xi,\alpha} L = d i_\xi L \,,$$
$$\delta_{\xi,\alpha} I_{CS} = d i_\xi I_{CS} + d\Xi_{\xi,\alpha} \,,$$

where

$$\Xi_{\xi,\alpha} = \alpha \frac{\partial I_{CS}}{\partial A} + \Lambda \frac{\partial I_{CS}}{\partial \mathbf{\Gamma}}$$

and, as before, $(\Lambda)^a_b = \partial_b \xi^a$. The new contribution appears as a total derivative. Therefore, it is still possible to define a Noether current

$$J_{\xi,\alpha} = \theta_{\xi,\alpha} - i_\xi \left(L + I_{CS}\right) - \Xi_{\xi,\alpha} \,,$$

that is closed on-shell

$$dJ_{\xi,\alpha} \doteq 0 \,.$$

Inspired by Section 3.2.2, we can define an improved current $\hat{J}_{\xi,\alpha}$ that vanishes on-shell through the addition of a total derivative. The interest of defining $\hat{J}_{\xi,\alpha}$ is that, according to the case without anomalies, it can be used to find conserved fluxes once ξ or α are chosen to preserve the solutions. In this case, $\Xi_{\xi,\alpha}$ introduces some subtleties because it is proportional to the gauge parameters instead of their derivatives. Once this is solved, though, most of the construction is essentially equal.

This issue can be circumvented introducing a $(d-1)$-form y such that

$$\Xi_{\xi,\alpha} - dy = \Xi'_{d\alpha,d\Lambda} \,, \tag{3.37}$$

where Ξ' only depends on the gauge parameters through their exterior derivative. Please notice this is not an on-shell identity, both sides must be equal independently of the equations of motion. The existence of a solution to this equation is conveniently guaranteed by the closedness of the anomaly polynomials. Rearranging the on-shell vanishing current, we get

$$\hat{J}_{\xi,\alpha} = \theta_{\xi,\alpha} - i_\xi (L + I_{CS}) - \Xi'_{d\alpha,d\Lambda} + dk'_{\xi,\alpha} \,, \tag{3.38}$$

where

$$k'_{\xi,\alpha} = k_{\xi,\alpha} - y$$

is the new candidate for the Komar form.

The importance of making the first three terms of the on-shell vanishing current (3.38) be proportional to $d\alpha$ and $d\Lambda$ becomes evident now. We can choose $d\alpha = d\Lambda = 0$ and cancel that part of $\hat{J}_{\xi,\alpha}$, subsequently obtaining the necessary closure relation

$$dk'_{\xi,\alpha} = 0 \,.$$

More details on the generality and coordinate dependence of this construction can be found in [29]. We now treat separately the cases of gauge transformations and diffeomorphisms.

Anomalous gauge charge

The gauge contribution to the presymplectic current can be obtained combining the presymplectic current of 3.2.3 with (3.20). It can be written as

$$\theta^{\mu}_{\alpha} = 2\left(\frac{\partial L}{\partial F_{\mu\nu}} + \frac{\partial I_{CS}}{\partial F_{\mu\nu}}\right)\partial_{\nu}\alpha\,,$$

while the anomalous term is given by

$$\Xi^{\mu}_{\alpha} = \alpha\frac{\partial I_{CS}}{\partial A_{\mu}}\,,$$

and the Noether charge is defined as the sum of both contributions

$$J_{\alpha} = \theta_{\alpha} - \Xi_{\alpha}\,.$$

We must look now for a surface term $k^{\mu\nu}_{\alpha}$ such that if we add its derivative $\nabla_{\nu}k^{\mu\nu}_{\alpha}$ to the Noether current, the latter vanishes on-shell. The result is

$$k^{\mu\nu}_{\alpha} = -2\alpha\left(\frac{\partial L}{\partial F_{\mu\nu}} + \frac{\partial I_{CS}}{\partial F_{\mu\nu}}\right)\,,$$

and the new Noether current reads

$$\hat{J}^{\mu}_{\alpha} = \alpha\left(-2\nabla_{\nu}\frac{\partial L}{\partial F_{\mu\nu}} - 2\nabla_{\nu}\frac{\partial I_{CS}}{\partial F_{\mu\nu}} - \frac{\partial I_{CS}}{\partial A_{\mu}}\right) = -\alpha\left(2\nabla_{\nu}\frac{\partial L}{\partial F_{\mu\nu}} + \Sigma^{\mu}_{A}\right)\,.$$

This last expression vanishes on-shell according to (3.18).

We still need to rearrange the Noether current, such that the dependence on the gauge parameter of all the terms apart from the Komar charge is only through its derivative. Let us particularize (3.37) to our case and express the right-hand side with the explicit dependence on $\nabla_{\mu}\alpha$ to give

$$\alpha\frac{\partial I_{CS}}{\partial A_{\mu}} - \nabla_{\nu}y^{\mu\nu} = \Xi'^{\mu\nu}\nabla_{\nu}\alpha\,.$$

From the structure of Chern-Simons terms, we know that there exists a quantity $\ell^{\mu\nu}$ that satisfies

$$\frac{\partial I_{CS}}{\partial A_{\mu}} = \frac{\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau}F_{\nu\rho}F_{\sigma\tau} = \nabla_{\nu}\ell^{\mu\nu}\,.$$

In particular, the value of this quantity is

$$\ell^{\mu\nu} = \frac{2\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau}A_{\rho}F_{\sigma\tau}$$

Using this, we can fix the form y in the gauge case to be

$$y^{\mu\nu} = \alpha\ell^{\mu\nu}\,.$$

For the Chern-Simons action in (2.24), one gets

$$\frac{1}{2}J^a_A = n_\mu \ell^{\mu\nu} P^a_\nu \,.$$

At this point, one might find intriguing that this term is precisely the one required to define a conserved current from a consistent current, according to (3.30).

The complete Komar charge can be written as

$$k'^{\,\mu\nu}_{\alpha} = -2\left(\frac{\partial L}{\partial F_{\mu\nu}} + \frac{\partial I_{CS}}{\partial F_{\mu\nu}}\right) - \ell^{\mu\nu}\,,$$

and it indeed gives rise to the conserved current

$$n_\mu k'^{\mu\nu} P^a_\nu = -J^a_{cons}\,.$$

As already discussed for the non anomalous case, if the fields decay fast enough at the boundary of Σ, the closedness of k'_α makes the flux of the gauge charge on Σ radially conserved. In the context of the holographic RG, this conservation for the flux can be reinterpreted as the conservation of the flux of the conserved membrane current J^a_{cons}

$$\partial_{\hat{r}} \int_\Sigma d^d x \sqrt{-\gamma} J^a_{cons} = 0\,.$$

We commented above that the Wald construction could be followed for anomalous cases but only at the expense of ending up with a noncovariant Komar charge. The noncovariance becomes now obvious when we present the result in terms of the noncovariant conserved current J^a_{cons}. This is not a major obstruction for our goal, since we will be able anyway to recover the gravitational anomalous transport from horizon fluctuations. However, for particular cases the continuity equation of the Komar form does not simply reduce to the radial conservation of the flux of the conserved membrane currents, due to the appearance of surface contributions. In those cases the extraction of the transport coefficients becomes more cumbersome.

In particular, as we discuss below, those surface contributions appear for the $U(1)^3$ anomaly. It was shown by [60] that it is a consequence of having a constant magnetic field present. In that same reference, the authors show that the results for the anomalous conductivities remain valid in a large number of theories in which the field strength appears in higher powers. From our Komar charge construction the difference between those theories would be introduced through the $\partial L_{mat}/\partial F$ contribution and we would expect all those terms to vanish at the horizon in the absence of external electric fields once infalling boundary conditions were imposed.

Another point made in [60] is that massive vector fields in the bulk, possibly in the form of a Stückelberg field, alter the form of the conservation equation and the conductivities receive nontrivial bulk corrections. It can be seen at the level of the equation of motion because the Maxwell equation is no longer a total derivative and, therefore, the bulk gauge field obtains an effective mass and the dual current acquires an anomalous dimension, ceasing to be conserved. In our construction, this behavior can be seen because, in the presence of a Stückelberg field θ, there is no choice for the gauge parameter that leads to an invariant transformation, since

$\delta_\alpha \theta = \alpha \neq 0$. Thus, there is no way to make the presymplectic form vanish on-shell and the Komar form fails to be closed. One particular case in which this happens will be discussed in Chapter 5. In that example, we interpret the corrections to the axial conductivity due to charged scalar fields as an infrared screening of the axial charge.

Anomalous diffeomorphism charge

The construction in the case of diffeomorphisms is slightly more involved due to the higher number of terms we need to take care of. We go on assuming mixed anomalies to lie in the diffeomorphism sector, according to (2.24), because the gravitational sector then depends only on field strengths and behaves better in the presence of constant background magnetic fields. The gauge sector, though, presents in this case some subtleties due to the asymptotic divergence of the vector potential A_μ.

The presymplectic current θ_ξ can be obtained from the Einstein-Maxwell part in (3.14), which we call here θ_ξ^{EM}, and the Chern-Simons contribution from (3.20). The final result is

$$\begin{aligned}\theta_\xi =&\theta_\xi^{EM} + \frac{\partial I_{CS}}{\partial F}\left(d i_\xi A + i_\xi F\right) + \frac{\partial I_{CS}}{\partial \mathbf{R}}\left(d i_\xi \mathbf{\Gamma} + i_\xi d\mathbf{\Gamma} + d\Lambda + [\Lambda, \mathbf{\Gamma}]\right)\\ &+ \Sigma^{\alpha\beta\mu}\nabla_{(\alpha}\xi_{\beta)} \star dx^\mu \,.\end{aligned} \tag{3.39}$$

The anomalous term is

$$\Xi_\xi = \Lambda \frac{\partial I_{CS}}{\partial \mathbf{\Gamma}}\,,$$

which supplements the presymplectic current to give the Noether current.

We need to add a total derivative dk_ξ to the Noether current

$$\hat{J}_\xi = \theta_\xi - i_\xi\left(L + I_{CS}\right) - \Xi_\xi + dk_\xi\,, \tag{3.40}$$

such that it makes the current vanish on-shell. Similarly to the presymplectic current, which we decomposed in an Einstein-Maxwell part and a Chern-Simons part, we introduce the notation that k_ξ has a part denoted by k_ξ^{EM} that is equal to (3.15) and a part stemming from the Chern-Simons action that needs to be worked out.

The explicit expression of the Hodge dual of k_ξ is, in components,

$$k_\xi^{\mu\nu} = (k_\xi^{EM})^{\mu\nu} - \frac{1}{2}\xi^\rho\left(\Sigma^\mu{}_\rho{}^\nu + \Sigma^{\nu\mu}{}_\rho + \Sigma_\rho{}^{\mu\nu}\right) + \xi^\rho A_\rho \frac{\partial I_{CS}}{\partial F_{\mu\nu}} + \nabla_\rho \xi^\sigma \frac{\partial I_{CS}}{\partial R^\sigma{}_{\rho\mu\nu}}\,. \tag{3.41}$$

Regarding the vanishing of the Noether current, it works as follows. On one hand, the first anomalous term in (3.41) combines with the last term in (3.39) to give the spin current. On the other hand, the rest of (3.41) combines with the rest of the Chern-Simons presymplectic current to give Maxwell's equations, after some manipulations using identities between differential forms.

Following the previous discussion for the gauge case, we add and subtract to (3.40) the total derivative of $y^{\mu\nu}$. If we find a $y^{\mu\nu}$ that makes the dependence on the gauge parameter of the rest of the Noether current be only through derivatives, we will have the on-shell conserved quantity

$$k'^{\mu\nu} = k^{\mu\nu} - y^{\mu\nu} .$$

We thus need to fix $y^{\mu\nu}$. It can be done once a particular theory is picked, as discussed in [29]. We take again the theory to be given by (2.24), so the anomaly is taken to be in the diffeomorphism sector. However, the search for the Komar charges is an on-shell construction and the equations of motion do not change under the addition of total derivatives to the Lagrangian, so this choice has no physical significance. One might wonder why we have insisted on this choice in all the discussion when in the literature about anomalous transport the anomaly is always taken in the gauge sector.

The reason why we choose this anomaly becomes evident in the next step, in which we construct the $y^{\mu\nu}$ for our particular theory. If we chose a Chern-Simons action like (2.22), $y^{\mu\nu}$ would be zero, but the last term in (3.41) would give a gauge dependent contribution

$$\nabla_\rho \xi^\sigma \frac{\partial I_{CS}}{\partial \mathbf{R}^\sigma{}_\rho} \sim \epsilon^{\mu\nu\alpha\beta\gamma} A_\gamma R^{\sigma\rho}{}_{\alpha\beta} \nabla_\rho \xi_\sigma .$$

This form is troublesome for the constant magnetic field analysis required to recover the anomalous transport coefficients. The linear coordinate dependence of a gauge connection of the form $A_i \propto \epsilon_{ijk} x^j B^k$ would make manipulations through Stokes theorem subtle and ill-defined.

There will only be contribution to Ξ from the mixed anomaly, which can be expressed schematically as

$$I_{CS} = \lambda F \wedge CS(\mathbf{\Gamma}) .$$

For this choice,

$$\frac{\partial I_{CS}}{\partial \mathbf{\Gamma}} = \lambda F \wedge \left(\mathbf{R} - \mathbf{\Gamma}^2\right) = \lambda F \wedge d\mathbf{\Gamma} ,$$

and the relation for y now reads

$$\Lambda \frac{\partial I_{CS}}{\partial \mathbf{\Gamma}} - dy = \Xi'_{d\Lambda} ,$$

so it becomes evident that y can be

$$y = \lambda \Lambda F \wedge \mathbf{\Gamma} ,$$

whose explicit expression in components is

$$y^{\mu\nu} = \lambda \epsilon^{\mu\nu\rho\tau\sigma} F_{\tau\sigma} \Gamma^\alpha_{\rho\beta} \Lambda^\beta_\alpha .$$

The final Komar charge is given by $k' = k - y$, whose flux over each constant radius hypersurface can be related to the RG equation for the membrane heat current. We need to take a static solution, for which $\xi = \partial_t$ is a Killing field, and compare the expression for this charge to the membrane stress tensor Θ^{ab}. Let us take a look at the explicit form of $n_\nu (k'_\xi)^{\mu\nu} P^a_\mu$ after ADM decomposing it. We express it separated in different terms that match the different contributions of the membrane currents, as in

$$
\begin{aligned}
\left(k^{\mu\nu}_{EH} + \xi^\rho A_\rho \frac{\partial I_{CS}}{\partial F_{\mu\nu}}\right) n_\nu P^a_\mu &= -\frac{1}{2} t^{ab}_{EH}\xi_b - \frac{1}{2} A^b \xi_b \mathcal{J}^a \,, \\
-\frac{1}{2}\xi_b \left(\Sigma^{\mu b\nu} + \Sigma^{b\mu\nu}\right) n_\nu P^a_\mu &= -\frac{1}{2} t^{ab}_\lambda \xi_b \,, \\
\left(-\frac{1}{2}\xi_b \Sigma^{\nu\mu b} + \nabla_\rho \xi^\sigma \frac{\partial I_{CS}}{\partial R^\sigma{}_{\rho\mu\nu}} - y^{\mu\nu}\right) n_\nu P^a_\mu &= -\frac{1}{2} u^{ac} K_{cb}\xi^b - \frac{1}{2}\zeta^a \,,
\end{aligned}
$$

where ζ^a is given by

$$
\zeta^a = 4 l^{(ab)}{}_c \Lambda^c_b + l^{ab}{}_c \hat{\Gamma}^c_{bd}\xi^d + l^{ba}{}_c \hat{\Gamma}^c_{bd}\xi^d - l^{bc}{}_d \hat{\Gamma}^a_{bc}\xi^d - 2\frac{1}{\sqrt{-\gamma}}\partial_b\left(\sqrt{-\gamma} l^{ba}{}_c \xi^c\right) .
$$

The equations above look rather messy. However, we can use the intuition we have gained from the $U(1)$ case to check whether they combine into the conserved heat current

$$
k^{a\hat{r}}_\xi = -\frac{1}{2}\left(\Theta^{ab}\xi_b + A^b\xi_b\mathcal{J}^a - \frac{1}{\sqrt{-\gamma}}\partial_b\left(\sqrt{-\gamma} l^{ba}{}_c\xi^c\right) + 2 l^{(ab)}{}_c \Lambda^c_b\right) . \tag{3.42}
$$

The computation is tedious but conceptually straight-forward, as one needs to compute explicitly the intrinsic contributions to Θ^{ab}. The computation simplifies greatly by realizing

$$
l^{ba}{}_c \hat{\Gamma}^c_{bd}\xi^d = -l^{bc}{}_d \hat{\Gamma}^a_{bc}\xi^d \,.
$$

One finally finds

$$
\Theta^{ab}\xi_b = t^{ab}_0\xi_b + t^{ab}_\lambda\xi_b + u^{ac}K_{cb}\xi^b - \frac{1}{\sqrt{-\gamma}}\partial_b\left(\sqrt{-\gamma} l^{ba}{}_c\xi^c\right) + 2 l^{(ab)}{}_c D_b\xi^c - l^{bc}{}_d \Gamma^a_{bc}\xi^d \,,
$$

which allows to prove (3.42).

Then, for surface terms decaying fast enough, the conservation equation reads

$$
\begin{aligned}
0 &= \partial_{\hat{r}} \int_\Sigma d^d x \sqrt{-\gamma}\left(\Theta^{ab}\xi_b + A^b\xi_b\mathcal{J}^a - \frac{1}{\sqrt{-\gamma}}\partial_b\left(\sqrt{-\gamma} l^{ba}{}_c\xi^c\right) + 2 l^{(ab)}{}_c\Lambda^c_b\right) \\
&= \partial_{\hat{r}} \int_\Sigma d^d x \sqrt{-\gamma} Q^a_{cons} \,.
\end{aligned}
$$

This must be interpreted as an RG equation for the membrane conserved heat current. Here, again, it is Θ^{ab} the one that appears in the RG equation and not the Brown-York tensor.

3.4 Extraction of anomalous transport coefficients

3.4.1 Asymptotics, thermodynamics, sources and radial conservation

Let us consider a generic asymptotically AdS stationary black hole solution. We must understand it as an order by order fluid-gravity expansion in spatial derivatives. The transport coefficients correspond to the first order corrections to the one point functions of the currents. However, it is instructive to keep the discussion as general as possible in the near-horizon region, where only regularity and stationarity need to be imposed in order to evaluate the extrinsic membrane currents. We are interested in the effects of both $U(1)^3$ and mixed gauge-gravitational anomalies. Therefore, we study charged black hole solutions with no electric fields.

Our ansatz now takes the form

$$\begin{aligned} ds^2 &= \frac{dR^2}{F(R)} - f(R)u_a u_b dx^a dx^b + g(R)h_{ab}dx^a dx^b\,, \\ A &= A_t(R)u_a dx^a + a_b dx^b\,, \end{aligned}$$

where we have naturally decomposed the metric and the gauge field into their projections parallel and perpendicular to the Killing vector $\xi^a = (1, \vec{0})$. In particular, we get

$$u_a = -\frac{1}{f(R)}\gamma_{ab}\xi^b\,, \quad A_t = A_a\xi^a\,,$$

and

$$h_{ab}\xi^b = h_a{}^b u_b = a_b\xi^b = 0\,.$$

All of the functions in the ansatz above, if not indicated explicitly, depend on the radial and the spatial coordinates only.

We no longer use the Fefferman-Graham coordinates near the horizon and we choose the new radial coordinate R in such a way that one can consistently impose the gauge fixing $A_R = 0$. In particular, we define R implicitly through the asymptotic expansions at the conformal boundary and the horizon because we will only evaluate the quantities involved in those regions. Near the boundary, the expansion is equal to the Fefferman-Graham one: $F(R) \sim 1$, $f(R) \sim g(R) \sim e^{2R}$ and all the rest of the quantities in the ansatz are of order one. At the horizon, asymptotics are fixed by regularity and infalling conditions [82]. In particular, $f(R)$ and $F(R)$ vanish at the location R_H of the horizon, while their quotient remains finite. Additionally, the derivative of $f(R)$ at the horizon is related to the Hawking temperature T through

$$\partial_R f = 4\pi T\sqrt{\frac{f}{F}} + O(f, F)\,.$$

This temperature must be interpreted as the temperature of the dual thermal state. At the horizon we can finally fix, without loss of generality, $g(R_H) = 1$ and $\partial_R g(R_H) = 1$, so that h_{ab} plays the role of the induced metric on the horizon.

A notion of chemical potential is also required to fully specify the thermodynamics of the system. In order to avoid cluttering of formulas, we fix the gauge for the vector potential in such a way that $A_t(R \to \infty) = 0$ and $A_t(R_H) = -\mu$, where μ is the chemical potential of the dual CFT state. However, as noncovariant charges are involved one may be worried by the fact that the choice of gauge could affect the result. The safe way to proceed would be erasing the dependence on gauge choice by working with the covariant version of the membrane current and stress tensor. In fact, we can always obtain an expression of the one-point function of the covariant currents in terms of horizon data even if the holographic RG equations are written in terms of the conserved version of the membrane currents. The RG equation can be written in a manifestly covariant way through the use of the gauge invariant chemical potential

$$\mu = \int dR\, F_{Ra}\xi^a = A_t(\infty) - A_t(R_H)\,.$$

Thus, we use that particular gauge for convenience but it has been checked that one could reproduce the result in a fully covariant way.

The magnetic field B^a and the vorticity ω^a are R-independent quantities that can be computed on each spacetime slice Σ as

$$\begin{aligned} B^a &= \sqrt{-\gamma}\epsilon^{abcd}u_b\partial_c a_d\,, \\ \omega^a &= -\frac{1}{2}\sqrt{-\gamma}\epsilon^{abcd}u_b\partial_c u_d\,. \end{aligned}$$

It should be noted that from this definition the curl of the $U(1)$ gauge field does not only include the magnetic field. The latter appears supplemented with a term proportional to the time component of the gauge field and the vorticity, according to

$$\sqrt{-\gamma}\epsilon^{abcd}u_b\partial_c A_d = B^a - 2A_t\omega^a\,. \tag{3.43}$$

The radial conservation equations vary under the change of coordinates. The definitions of the conserved fluxes, (3.9) and (3.10), do not vary, thanks to the appearance of the Levi-Civita tensor. However, one must be careful when performing the radial integration to include appropriate factors of the normal vector n_μ. For simplicity, we write the conservation equation as

$$\partial_R I^a + \int_\Sigma d^4x\partial_b\left(\sqrt{-\gamma}\frac{1}{\sqrt{F}}k_\alpha^{ba}\right) = 0\,, \tag{3.44}$$

$$\partial_R H^a + \int_\Sigma d^4x\partial_b\left(\sqrt{-\gamma}\frac{1}{\sqrt{F}}k_\xi^{ba}\right) = 0\,. \tag{3.45}$$

Now the relations between boundary and horizon data are obtained by integrating in dR instead of $d\hat{r}$.

Infalling boundary conditions at the horizon are typically imposed in order to compute the appropriate retarded correlators. In computations involving response to constant magnetic field or constant vorticity, like ours, these conditions play no role because the transport coefficients could be obtained from zero-frequency correlators. However, they might be important, for example, if we computed response to electric fields. In our setup, though, they still prove useful, since they guarantee the vanishing of the nonanomalous part of the current. We now show this vanishing in a few lines.

Let us introduce ingoing and outgoing Eddington-Finkelstein coordinates

$$dv = u_a dx^a + \frac{dR}{\sqrt{fF}}\,, \qquad du = u_a dx^a - \frac{dR}{\sqrt{fF}}\,.$$

Infalling boundary conditions reduce in this set of coordinates to independence of u at the horizon. In particular, for the field strength of the gauge field one gets

$$F_{ua} = \xi^b F_{ba} - \sqrt{fF} F_{Ra} = 0\,.$$

From here, we obtain that

$$\xi^b F_{ba} = \sqrt{fF} F_{Ra} = \sqrt{\frac{f}{F}} F^R{}_a = \sqrt{f} n_\mu F^\mu{}_a\,,$$

and, subsequently,

$$\int_H d^4x \sqrt{-\gamma} n_\mu F^{\mu b} = \int_H d^4x \sqrt{-\gamma} \frac{1}{\sqrt{f}} \xi^a F_a{}^b\,,$$

which is finite and proportional to the electric field's flux at the horizon. Then, in our setup, infalling boundary conditions, or u-independence at the horizon, imply the vanishing of the fluxes of F^{Ra} at the horizon.

It was advanced above in Section 3.3.2 that the extrinsic curvature is expected to play a crucial role in the extraction of the transport properties. It becomes evident from its definition, which for our choice of coordinates reads

$$K_{ab} = \frac{\sqrt{F}}{2} \partial_R \gamma_{ab}\,.$$

We can then take its expansion near the horizon

$$K_{ab} = \frac{1}{2}\sqrt{f}\left(-4\pi T u_a u_b + h_{ab}\right) + O(f, F)\,.$$

The extrinsic curvature is thus our candidate for introducing temperature factors in the transport coefficients.

3.4.2 Membrane paradigm for anomalous currents

We finally compute the transport coefficients making use of all the mathematical tools introduced in the previous sections. The structure followed will be presenting the conserved fluxes, expressing them in terms of the membrane currents and then performing the computations of the one-point functions of the covariant operators. We treat the two types of anomalies separately. First, we analyze the mixed anomaly and, then, we move on to cover the $U(1)^3$ anomaly.

Mixed gauge-gravitational anomaly

The precise form of the Chern-Simons action considered is

$$I_{CS} = \int d^5x\sqrt{-g}2\lambda\epsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}\left(\Gamma^{\alpha}_{\rho\beta}\partial_\sigma\Gamma^{\beta}_{\tau\alpha} + \frac{2}{3}\Gamma^{\alpha}_{\rho\gamma}\Gamma^{\beta}_{\sigma\alpha}\Gamma^{\gamma}_{\tau\beta}\right) .$$

Since we add no Bardeen counterterm, this corresponds to the choice $c = 0$ in the construction of the consistent membrane currents from Section 3.3.2. One can explicitly verify that, with this choice, the continuity equations given by (3.44) and (3.45) can be integrated on Σ with no contributions from the boundary, so both conserved fluxes, I^a and H^a, will be radially conserved. These fluxes have already been shown to be respectively equivalent to the fluxes of the conserved versions of the current and heat current

$$I^a = -\int_\Sigma d^dx\sqrt{-\gamma}J^a_{cons} , \tag{3.46}$$

$$H^a = \frac{1}{2}\int_\Sigma d^dx\sqrt{-\gamma}Q^a_{cons} . \tag{3.47}$$

The explicit expressions for the integrands are

$$\begin{aligned} J^a_{cons} =& \mathcal{J}^a = \frac{1}{\sqrt{F}}F^{Ra} + J^a_{CSK} + J^a_{\hat{\Gamma}} , \\ Q^a_{cons} =& \Theta^a{}_b\xi^b + A_c\xi^c\mathcal{J}^a - \frac{1}{\sqrt{-\gamma}}\partial_b\left(\sqrt{-\gamma}l^{ba}{}_c\xi^c\right) + 2l^{(ab)}{}_c\Lambda^c_\xi . \end{aligned}$$

When integrating the radial conservation equations, we will use the fact that the evaluation of the fluxes on the conformal boundary give rise to the integrals of the CFT one-point functions of the respective operators.

Let us start with the current one-point functions. Integrating in dR the conservation equation for the gauge flux $\partial_R I^a = 0$ gives

$$\int d^4x\sqrt{-\gamma_{(0)}}\langle J^a_{cons}\rangle = \int_H d^4x\sqrt{-\gamma}J^a_{cons}(R_H) .$$

Since we are interested in contributions up to first order in derivative, the left-hand side reduces to the integral of the one-point function of the covariant current because the term $J^a_{\hat{\Gamma}}$ gives contributions starting from third order. In the horizon evaluation, we can additionaly discard the non-anomalous F^{Ra} term from the discussion above about the infalling boundary conditions. Therefore, the one point function of the covariant current can be expressed as the horizon integral of the extrinsic Chern-Simons current J^a_{CSK}

$$\int d^4x\sqrt{-\gamma_{(0)}}\langle J^a\rangle = \int_H d^4x\sqrt{-\gamma}J^a_{CSK}(R_H) + O(\partial^3) .$$

The value of the Chern-Simons current at the horizon is completely general and is only a consequence of the regularity of the near horizon geometry. In fact, we can obtain the final result explicitly expanding its expression

$$\begin{aligned}\sqrt{-\gamma}J^a_{CSK} &= -8\lambda\sqrt{-\gamma}\epsilon^{abcd}K_b{}^eD_cK_{de}\\ &= 32\lambda\pi^2T^2\sqrt{-\gamma}\epsilon^{abcd}u_b\partial_cu_d + \lambda\sqrt{-\gamma}\epsilon^{abcd}\left(16\pi Tu_bu^eD_ch_{de} - 2fh_b{}^eD_ch_{de}\right)\\ &= 32\lambda\pi^2T^2\sqrt{-\gamma}\epsilon^{abcd}u_b\partial_cu_d + O(f,F)\,.\end{aligned}$$

Integrating and remembering that the vorticity is constant along the radial coordinate, one gets the familiar answer

$$\int d^4x\sqrt{-\gamma_{(0)}}\langle J^a\rangle = -64\lambda\pi^2T^2\omega^a\,.$$

We stress how the gravitational transport follows from the state dependent extrinsic contribution to the membrane currents given by J^a_{CSK}. It would be interesting to understand whether it is possible to link out of equilibrium fluctuations of the chiral vortical effect to horizon variations of such membrane current.

Let us now turn to the treatment of the diffeomorphism conserved flux. We are interested in expanding only to first order in derivatives. Therefore, if both the horizon and the boundary are flat, we drop all the intrinsic contributions involving l^{abc} because their first contribution appears at third order in derivatives. Then, integrating the diffeomorphism flux H^a between the boundary and the horizon gives

$$\int d^4x\sqrt{-\gamma_{(0)}}\langle T^a{}_b\xi^b\rangle = \int_H d^4x\sqrt{-\gamma}\left(t^a{}_b\xi^b + u^{ac}K_{cb}\xi^b + A_c\xi^cJ^a_{CSK}\right) + O(\partial^3)\,,$$

where we have implicitly used the gauge choice $A(R\to\infty)=0$ and substituted the current in the horizon by the only nonvanishing contribution at that point J^a_{CSK}.

We need to evaluate the right-hand side on the horizon to extract the one-point function of the energy current. The third term on the right-hand-side follows immediately from the evaluation of the gauge fluxes and gives

$$\int_H d^dx\sqrt{-\gamma}A_c\xi^cJ^a_{CSK} = 64\lambda\pi^2T^2\mu\omega^a\,,$$

where we have used the gauge choice $A_c\xi^c(R_H) = A_t(R_H) = -\mu$. The first term of the right-hand-side, on the other hand, can be split into the Einstein-Hilbert part $(t_{EH})^a{}_b$ given in (3.12) and the anomalous part $(t_\lambda)^a{}_b$, which is made up of spin currents and given in (3.32). It can be shown that the former gives the ideal part of the stress tensor [33], which however does not contribute to the heat current due to the orthogonal projection. The anomalous part can also be shown to vanish at the horizon

$$\sqrt{-\gamma}t^a_{\lambda b}\xi^b(R_H) = 32\pi^2T^2\lambda\sqrt{-\gamma}\epsilon^{efga}\left(u_fu_b\xi^b - u_fu_b\xi^b\right)F_{ge} + O(f,F) = 0\,,$$

where we have used the asymptotic expansion of the extrinsic curvature and the vanishing of field-strength at the horizon. Please notice that these two quantities correspond to the Brown-York prescription for the stress tensor at the horizon, although they show no anomalous transport. The inclusion of the operator u^{ab}

sourced by the extrinsic curvature in the definition of the membrane stress tensor becomes thus necessary. The evaluation of this contribution gives

$$\sqrt{-\gamma}u^{ab}K_{bc}\xi^{c} = 16\pi^2T^2\lambda\epsilon^{aefg}F_{ef}u_g + O(f,F) = 32\pi^2T^2\lambda\left(B^a + 2\mu\omega^a\right) .$$

Expressing everything together, we obtain

$$\int d^4x\sqrt{-\gamma_{(0)}}\langle T^a{}_b\xi^b\rangle = 32\pi^2T^2B^a + 128\pi^2T^2\mu\omega^a .$$

which gives the right transport coefficients for the anomalous gravitational transport. It must be noted that now the chiral magnetic effect for the energy current, which persists without chemical potential, is completely captured by the horizon fluctuations of the extrinsic part of the proposed membrane stress tensor $\Theta^a{}_b$. It could therefore be interpreted as the energy counterpart of J^a_{CSK} and it might be interesting to explore the possibility that further information about nonequilibrium dynamics of the energy chiral magnetic effect may be encoded in the time dependent fluctuations of this quantity.

U(1)^3 anomaly

We fix the Chern-Simons action to be

$$I_{CS} = \int d^5x\sqrt{-g}\frac{\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau} .$$

The important continuity equations are again given by (3.44) and (3.45), and the fluxes can also be connected for this case to the conserved versions of the current and heat current, according to (3.46) and (3.47). However, further care is needed when integrating the continuity equations, as noticed in [60]. The Komar charges now depend explicitly on A_a. This impedes a priori the application of Stokes theorem if constant magnetic fields are present. However, it is possible for our case to rearrange the expressions involving A_a as total radial derivatives, but we would not expect this feature in general.

The explicit expression of the quantities involved in the continuity equations, for the gauge choice $A_R = 0$, reads

$$\begin{aligned}
J^a_{cons} &= \frac{1}{\sqrt{F}}F^{Ra} + 2\kappa\epsilon^{abcd}A_bF_{cd} ,\\
k^{ba}_{\alpha} &= -4\kappa\sqrt{F}\epsilon^{abcd}A_c\partial_RA_d ,\\
Q^a_{cons} &= (t_{EH})^a{}_b\xi^b + \xi_cA^c\mathcal{J}^a ,\\
\mathcal{J}^a &= \frac{1}{\sqrt{F}}F^{Ra} + \frac{4\kappa}{3}\epsilon^{abcd}A_bF_{cd} ,\\
k^{ba}_{\xi} &= \frac{4\kappa}{3}\sqrt{F}\epsilon^{abcd}A_c\partial_RA_dA_e\xi^e ,
\end{aligned}$$

where we have already dropped from k^{ba}_{α} and k^{ba}_{ξ} the parts that decay sufficiently fast at the boundary of the slice and, therefore, give no contribution to the continuity equations. As we already advanced above, we can rearrange the new term in the closure relation of k_α as a total radial derivative. The derivation begins with

$$\begin{aligned}\int_\Sigma d^d x \partial_b \left(\sqrt{-\gamma}\frac{1}{\sqrt{F}}k_\alpha^{ba}\right) &= -\int_\Sigma d^d x \partial_b \left(\sqrt{-\gamma} 4\kappa \epsilon^{abcd} A_c \partial_R A_d\right) \\ &= -\int_\Sigma d^d x \sqrt{-\gamma} 4\kappa \epsilon^{abct} \left(\partial_b A_c \partial_R A_t - A_t \partial_R \partial_b A_c\right) .\end{aligned}$$

In order to obtain the expression in the second line we need to realize that a is a spatial index and there is no time dependence on any quantity, so either c or d are equal to 0, and, moreover, $\partial_b A_t = 0$. Then the expression in the second line can be explicitly expanded in all the possible terms. The first contribution gives

$$-\int_\Sigma d^d x \sqrt{-\gamma} 4\kappa \epsilon^{abct} \partial_b A_c \partial_R A_t = -\int_\Sigma d^d x \sqrt{-\gamma} 4\kappa \epsilon^{abct} \left(A_t \partial_R A_t \partial_b u_c + \partial_R A_t \partial_b a_c\right) , \tag{3.48}$$

and the second contribution gives

$$\int_\Sigma d^d x \sqrt{-\gamma} 4\kappa \epsilon^{abct} A_t \partial_R \partial_b A_c = \int_\Sigma d^d x \sqrt{-\gamma} 4\kappa \epsilon^{abct} \left(A_t \partial_R \left(A_t \partial_b u_c\right) + A_t \partial_R \partial_b a_c\right) . \tag{3.49}$$

The first term of (3.48) can be combined with the first term of (3.49) to give

$$\int_\Sigma d^d x \sqrt{-\gamma} 4\kappa \epsilon^{abct} A_t^2 \partial_R \partial_b u_d .$$

However, this last contribution and the second term in (3.49) are proportional to the radial derivative of the vorticity and the magnetic field, respectively, so both contributions vanish. Therefore, only the second term of (3.48) survives and it can be rewritten as a total radial derivative. The final result for the extra term in the radial conservation of the flux is

$$\int_\Sigma d^d x \partial_b \left(\sqrt{-\gamma}\frac{1}{\sqrt{F}}k_\alpha^{ba}\right) = -\partial_R \int_\Sigma d^d x \sqrt{-\gamma} 4\kappa A_t B^a .$$

The same reasoning applies to the diffeomorphism charge, thus giving

$$\int_\Sigma d^d x \partial_b \left(\sqrt{-\gamma}\frac{1}{\sqrt{F}}k_\xi^{ba}\right) = \partial_R \int_\Sigma d^d x \sqrt{-\gamma}\frac{2\kappa}{3} A_t^2 B^a .$$

The only difference in the procedure appears in the integration by parts. It now has to be performed carefully, realizing that $A_t \partial_R A_t = \partial_R (A_t^2/2)$.

Let us go back to the computation of the one-point functions. The gauge continuity equation (3.44) now reads

$$\partial_R \int_\Sigma d^4 x \sqrt{-\gamma} \left(J_{cons}^a + 4\kappa A_t B^a\right) = 0 .$$

Integrating in the radial direction we get

$$\int d^4 x \sqrt{-\gamma_{(0)}} \langle J_{cons}^a \rangle = \int_H d^4 x \sqrt{-\gamma} J_{cons}^a (R_H) - 4\kappa \mu B^a ,$$

where we have used the gauge choice $A_t(R_H) = -\mu$ and the definition of the magnetic field. We can now relate the conserved current to the covariant one

$$J^a_{cons} = J^a + 4\kappa\epsilon^{abcd} A_b \partial_c A_d \, .$$

At the horizon, the first term vanishes due to infalling boundary conditions and the second term gives $4\kappa A_t$ times (3.43), so we obtain

$$\int_H d^d x \sqrt{-\gamma} J^a_{cons}(R_H) = -4\kappa\mu \left(B^a + 2\mu\omega^a \right) \, . \tag{3.50}$$

On the other hand, the second term in the conserved current gives no contribution in the boundary due to our gauge choice $A_t(R \to \infty) = 0$. Therefore, the final result for the one-point function is

$$\int d^d x \sqrt{-\gamma_{(0)}} \langle J^a \rangle = -8\kappa\mu B^a - 8\kappa\mu^2 \omega^a \, ,$$

which fits the usual results for holographic anomalous transport coefficients.

The diffeomorphism continuity equation (3.45) now reads

$$\partial_R \int_\Sigma d^d x \sqrt{-\gamma} \left(Q^a_{cons} + \frac{4}{3}\kappa A_t^2 B^a \right) = 0 \, .$$

Since we are here considering only the $U(1)^3$ anomaly, the usual Brown York tensor obtained from the Einstein-Hilbert action is the only contribution to the membrane stress tensor and the complete conservation law is

$$\partial_R \int_\Sigma d^4 x \sqrt{-\gamma} \left(T^a{}_b \xi^b + A_c \xi^c \mathcal{J}^a + \frac{4}{3}\kappa A_t^2 B^a \right) = 0 \, .$$

In our gauge choice this gives the matching

$$\int d^4 x \sqrt{-\gamma_{(0)}} \langle T^a{}_b \xi^b \rangle = \int_H d^4 x \sqrt{-\gamma} A_c \xi^c \mathcal{J}^a + \frac{4}{3}\kappa\mu^2 B^a \, ,$$

where we have already discarded the contribution of the Brown-York tensor at the horizon. The evaluation of the consistent current is completely parallel to (3.50). The final result is

$$\int d^4 x \sqrt{-\gamma_{(0)}} \langle T^a{}_b \xi^b \rangle = -\int d^4 x \sqrt{-\gamma_{(0)}} \langle J^a_\epsilon \rangle = 4\kappa\mu^2 B^a + \frac{16}{3}\kappa\mu^3 \omega^a \, ,$$

where we have also introduced the energy current. The result matches the usual equilibrium values.

3.5 Discussion

We have extended the construction of membrane currents to anomalous theories. In doing so we identified how extrinsic contributions coming from gravitational

Chern-Simons terms are linked to the thermal anomalous transport at the horizon. Such terms vanish at the conformal boundary but are dynamically generated at lower energies, finally giving the expected thermal effective action on the horizon. This is very reminiscent of the Wilsonian integration of gapped excitations. Thus, it would be interesting to see if such a parallel can indeed be made and, in that case, how those modes have to be interpreted.

The horizon properties can be reformulated as CFT observables through the usage of conserved fluxes and we have shown that such fluxes coincide, up to subtleties in the $U(1)^3$ case, with *conserved* membrane fields. This is a nontrivial extension of the previous arguments, allowing us to explain various results found in the literature in a simple and elegant way.

Finally, holographic systems have long been used in the study of non-equilibrium processes and the first studies regarding the gravitational anomaly have been recently published [106]. It would be interesting to see if the membrane currents we have defined, which precisely account for these anomalous hydrodynamic fluctuations at the horizon, could be used to get analytical insight over such phenomena.

Chapter 4

Anomalous transport and holographic momentum relaxation

The currents associated to the chiral magnetic and vortical effects are dissipationless. In this chapter our goal will be studying if these anomalous transport phenomena are affected by the breaking of translation symmetry. This breaking is performed in holography via the inclusion of linear massless scalar field couplings such that the graviton acquires an effective mass. We show that the chiral magnetic and vortical conductivities are independent of the holographic disorder coupling. The model with momentum relaxation reproduces the usual equilibrium values of the conductivities in terms of chemical potential and temperature. However, it requires the use of the membrane stress tensor introduced in the previous chapter, which solves the puzzle found in [115] when using the uncorrected form of the energy-momentum tensor. One can thus understand this chapter as a corollary to the previous one.

This chapter is based on [38]. It is organized as follows: in Section 4.1, we motivate the project and comment on the specific purpose of the work; in Section 4.2, we present the 5-dimensional model of holographic momentum relaxation, we discuss the construction of the background and introduce the form of the current and energy-momentum tensor necessary to obtain the right transport coefficients; in Section 4.3, we compute the DC electric conductivity and also the chiral magnetic and chiral vortical conductivities; finally, in Section 4.4, we sum up the results and include some conclusions about the project.

4.1 Motivation

In the context of holography, higher dimensional black holes can be used to study strongly coupled quantum systems because the hydrodynamics of the latter can be mapped to the dynamics of the black hole horizon. In hydrodynamics, momentum is an exactly conserved quantity and its conservation produces convective transport of charge and infinite DC conductivities. However, in applications to condensed matter physics one would expect to find momentum conservation broken and, consequently, finite DC conductivities.

Thus, it is necessary to find a way to engineer momentum relaxation in holographic theories in order to apply the gauge/gravity duality to strongly coupled condensed matter systems. It can be done by making the graviton acquire an effective mass [145, 41, 27, 28, 42]. A particularly simple mechanism to generate the graviton mass, which we will use in the following, involves massless scalar fields with spatially linear profiles [9].

There have been several publications analyzing the models with massless scalars for field theories in (2+1) dimensions. In [61] it was found that the holographic DC conductivity σ satisfies a lower bound $e^2\sigma \geq 1$ in terms of the bulk Maxwell coupling e^2. This implies that this form of disorder cannot produce a transition between a metal phase and an insulator phase. However, one could in principle include additional couplings, resulting in the renormalization of e and, possibly, a change in the bound. In particular, [14, 59] show that coupling between the massless scalars and the gauge field gives rise to conductivities unbounded from below. Furthermore, for certain regions of parameter space the conductivities can become negative, signaling the appearance of instabilities.

We find intriguing the effect this form of holographic disorder could have on anomalous transport. For the sake of generality, we consider the models with further couplings between the scalar and gauge sectors used in [14, 59]. However, chiral anomalies only appear on even dimensions so some minor considerations must be taken into account in order to generalize their theory to (3+1) dimensions. On one hand, the electric DC conductivity will still be present for one higher dimension but the scaling of the conserved current and, as a result, of the conductivity changes. On the other hand, we need to introduce the anomalies via Chern-Simons terms and this will produce the well-known chiral magnetic and chiral vortical effects.

In particular, we include a $U(1)^3$ and a mixed gauge gravitational anomaly, whose associated Chern-Simons actions have already been extensively discussed in previous chapters. Since the chiral magnetic and chiral vortical effects are produced by the anomaly, one expects them to receive no quantum corrections and to give rise to dissipationless transport. Thus, in this chapter our purpose is to check explicitly that their form does not depend on the mass of the graviton or the disorder parameters.

4.2 Holographic momentum relaxation

The action of our model is

$$S = \int d^5x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R+\frac{12}{L^2}\right) - \frac{1}{2}\partial^\mu X^I \partial_\mu X^I - \frac{1}{4}F^2 - \frac{J}{4}\partial_\mu X^I \partial_\nu X^I F^\mu{}_\lambda F^{\lambda\nu}\right] + S_{CS} + S_{GH} + S_{CSK}\,, \tag{4.1}$$

where $I = 1, 2, 3$, G is Newton's constant, L is the AdS radius and we have already fixed the cosmological constant $\Lambda = -6/L^2$ that gives AdS spacetime in 5 dimensions. From now on we set $16\pi G = L = 1$.

Momentum relaxation is introduced in holography using a Stückelberg mechanism on the gravity sector. The three massless scalar fields X^I play the role then of Goldstone modes associated to the breaking of translation symmetry and, through a spatially linear profile, they give a mass to the graviton. We also include a Maxwell field A though the usual kinetic term of a gauge field (where $F = dA$) and the scalars couple to this gauge sector via the term proportional to J, which represents one of the possible interactions used to monitor the effects of disorder on the charged sector of the theory [14, 59]. We expect the gauge sector to be anomalous, receiving contribution both from a $U(1)^3$ and a mixed gauge-gravitational anomaly. Thus, we add a Chern-Simons action S_{CS} of the form (2.22), such that diffeomorphisms are not anomalous. The extra terms S_{GH} and S_{CSK} are local counterterms. The former is the Gibbons-Hawking term. It is given by (3.11) and is usually included to have a well-defined variational problem. The other counterterm S_{CSK} is given by (3.27) and is included to give the right anomalous Ward identity when the mixed anomaly is completely on the gauge sector, as discussed in Section 3.3.2.

The equations of motion are

$$
\begin{aligned}
0 =& G_{\mu\nu} - 6g_{\mu\nu} + \frac{1}{2}F_{\mu}{}^{\lambda}F_{\lambda\nu} - \frac{1}{8}g_{\mu\nu}F^2 - \frac{1}{2}\partial_\mu X^I \partial_\nu X^I + \frac{1}{4}g_{\mu\nu}\partial_\rho X^I \partial^\rho X^I \\
& - \frac{J}{4}\left(\tilde{X}.F.F + F.\tilde{X}.F + F.F.\tilde{X}\right)_{\mu\nu} + \frac{J}{8}g_{\mu\nu}\mathrm{Tr}\left(\tilde{X}.F.F\right) - 2\lambda\epsilon_{\alpha\beta\gamma\delta(\mu}\nabla_\rho\left(F^{\beta\alpha}R^{\rho}{}_{\nu)}{}^{\gamma\delta}\right) , \\
0 =& \nabla_\nu F^{\nu\mu} + \frac{J}{2}\nabla_\nu(\tilde{X}.F)^{\mu\nu} - \frac{J}{2}\nabla_\nu(\tilde{X}.F)^{\nu\mu} + \epsilon^{\mu\nu\rho\sigma\tau}\left(\kappa F_{\nu\rho}F_{\sigma\tau} + \lambda R^{\alpha}{}_{\beta\nu\rho}R^{\beta}{}_{\alpha\sigma\tau}\right) , \\
0 =& \Box X^I + \frac{J}{2}\nabla_\mu\left(\partial_\nu X^I F^{\nu}{}_{\lambda}F^{\lambda\mu}\right) ,
\end{aligned}
$$

where we have defined $\tilde{X}_{\mu\nu} = \partial_\mu X^I \partial_\nu X^I$ inspired by [59], since this allows us to introduce a very compact notation for contractions e.g. $\left(\tilde{X}.F.F\right)_{\mu\nu} = \partial_\mu X^I \partial_\rho X^I F^{\rho}{}_{\lambda}F^{\lambda}{}_{\nu}$.

As already commented, momentum dissipation is implemented by giving the scalar fields a linear profile. We associate each of the three scalars to one of the three spatial dimensions, and define

$$
X^1 = kx\,, \quad X^2 = ky\,, \quad X^3 = kz\,.
$$

Because the scalars couple only through derivatives, the field equations and solutions will still be formally translational invariant. The parameter k gives the graviton however a mass and this suffices to make the DC conductivity of a charged black hole solution finite. As we will see, it also fixes a preferred frame (the rest frame of the impurity density) and no frame ambiguity in the definition of the anomalous transport coefficients appears [6, 123, 132].

4.2.1 Background construction

In order to define the background we look for charged black hole solutions with AdS asymptotics of the form

$$
\begin{aligned}
ds^2 &= \frac{1}{\hat{u}}\left(-f(\hat{u})dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{d\hat{u}^2}{4\hat{u}^2 f(\hat{u})}, \\
A_t &= \phi(\hat{u})\,,
\end{aligned}
$$

where $\hat{u}$ relates to the usual Poincaré radial coordinate through $\hat{u} = 1/r^2$. Since all the functions only depend on $\hat{u}$, we will drop the dependences from the expressions and use primes in this chapter to denote derivatives with respect to $\hat{u}$. The equations of motion for the blackening factor and the time component of the gauge field are

$$\begin{aligned} 0 &= f' + \frac{2}{\hat{u}}(1-f) - \frac{\hat{u}^2}{3}(\phi')^2 - \frac{k^2}{4}, \\ 0 &= \phi''. \end{aligned}$$

The solution of the equations has three total integration constants: one for f and two for ϕ. The one in f is fixed by imposing that the horizon sits at $\hat{u} = 1$. The two integration constants in ϕ are fixed by imposing the definition of the chemical potential ($\mu = \phi(0) - \phi(1)$) and that ϕ vanishes at the horizon. The final solution reads

$$\begin{aligned} f &= (1-\hat{u})\left(1 + \hat{u} - \frac{k^2}{4}\hat{u} - \frac{\mu^2}{3}\hat{u}^2\right), \\ \phi &= \mu(1-\hat{u}). \end{aligned}$$

The temperature is

$$T = \frac{1}{\pi}\left(1 - \frac{k^2}{8} - \frac{\mu^2}{6}\right).$$

In this simple theory the background does not depend on the charge disorder coupling J and it therefore reduces to the solution found in [9].

4.2.2 Holographic covariant currents

As we have seen in the previous chapter, the mixed gauge-gravitational Chern-Simons term makes the analysis of the holographic dictionary much more complicated than in the standard case because it is a higher derivative term. The standard definitions of the holographic operators change even if we treat the anomalous action in an effective field theory spirit, computing only the leading effects of a expansion in small λ.

Let us now specialize the definitions of the holographic operators proposed in Section 3.3.2 to the current example. For convenience we switch in this section to the standard Fefferman-Graham coordinates, given by (3.1), so that we can recover all the expressions of the operators as they are written above. We will work with the covariant version of the operators.

Please note that, although we do not explicitly write the expression of the one-point functions of the currents to avoid cluttering, it involves the computation of integrals with the appropriate volume element, according to what we did in Section 3.4. The reader might be confused because this is not the standard way to define the current in holography. As it will be done in Chapters 5 and 6, the standard definitions of CFT operators in holography also include the square root of the determinant of the induced metric.

The covariant Ward identity for diffeomorphisms is

$$D_i\left(t^i{}_j + u^{il}K_{lj}\right) = u^{il}D_jK_{il} + F^{ij}J_i - Y^I D_j X_I - 2\lambda D_k\left(\epsilon^{lmnp}F_{lm}R^k{}_{jnp}\right),$$

where we have decomposed the covariant membrane stress tensor given in (3.34) in its two contributions and we have introduced

$$Y^I = \dot{X}^I + \frac{J}{2}\left(F^{\hat{r}i}\dot{X}^I + F^{ji}\partial_j X^I\right)F_{i\hat{r}}\,,$$

which is the momentum conjugate to X^I. Dots denote derivatives with respect to the Fefferman-Graham radial coordinate. The covariant current J^i also appears in the covariant Ward identity and now it receives a further contribution proportional to the J coupling, according to

$$J^i = F^{ir} - 8\lambda\epsilon^{ijkl}K_j^m D_k K_{lm} + \frac{J}{2}\left(F^{i\hat{r}}\partial_i X^I \partial^j X_I - \dot{X}_I\dot{X}^I F^{\hat{r}j} - \dot{X}_I\partial_i X^I F^{ij}\right)\,.$$

The covariant Ward identity for gauge transformations gives the usual anomalies

$$D_i J^i = -\epsilon^{ijkl}\left(\kappa F_{ij}F_{kl} + \lambda R^a{}_{bij}R^b{}_{akl}\right)\,.$$

Please notice the factor $1/3$ difference with respect to the consistent form of the anomaly, given by (3.26). In fact, one could use the holographic anomaly coefficients κ and λ and compare the results to those of a weakly coupled theory with N_χ chiral fermions. The matching of anomaly coefficients gives

$$\kappa = \frac{N_\chi}{32\pi^2}\,,\quad \lambda = \frac{N_\chi}{768\pi^2}\,. \tag{4.2}$$

The tensor t^{ij} can be divided into the standard Brown-York contribution t^{ij}_{EH} and a part that stems from the mixed gauge gravitational Chern-Simons term t^{ij}_λ, and our new definition of stress tensor also includes the other operator u^{ij}, as discussed in Section 3.3.2. Let us include them all here for the sake of clarity when we use them later

$$\begin{aligned}
t^{ij}_{EH} &= -2(K^{ij} - K\gamma^{ij}) \\
t^{ij}_\lambda &= -8\lambda\epsilon^{mnp(i}\left(2D_n K_p^{j)}F_{\hat{r}m} + \gamma^{j)l}\dot{K}_{ln}F_{pm} - F_{pm}K_l^{j)}K_n^l\right)\,, \\
u^{ij} &= 8\lambda\epsilon^{mnp(i}F_{mn}K_p^{j)}\,,
\end{aligned}$$

So far, the additional terms in the stress tensor might appear to be only required by our desire to have constraint equations that are formally equal to the usual field theory ones or to have nicely behaved holographic RG equations. However, we will see they are essential to get the physical results of the anomalous transport coefficients, thus generalizing the expression found in [117] to cases in which $\gamma^{(-2)}$ does not vanish in (3.2).

The usual contribution $(t_{EH})^i_j$ is still divergent and needs to be regularized by the standard counterterms [70]. In contrast, the additional contributions $(t_\lambda)^i_j$ and $u^{il}K_{lj}$ are already finite before the holographic renormalization is performed. We must remark that, in the case of holographic pure gravitational anomalies dual to two-dimensional field theories, a similar correction has been found in [99].

Once we have clarified which are the relevant operators, it is time to move on to study the linear responses. We will first compute the electric DC conductivity and then the chiral magnetic and chiral vortical effect.

4.3 Conductivities

4.3.1 Electric DC conductivity

Let us introduce the small perturbations

$$\begin{aligned} A_z &= \epsilon(-Et + a_z(\hat{u}))\,, \\ g_{tz} &= \frac{\epsilon}{\hat{u}} h_t^z(\hat{u})\,, \\ g_{\hat{u}z} &= \frac{\epsilon}{\hat{u}} h_{\hat{u}}^z(\hat{u})\,, \end{aligned}$$

where E is the external electric field and the perturbations do not introduce additional sources, i.e. they are zero at the conformal boundary. The equations of motion are

$$\begin{aligned} 0 &= \left(1 + 2J\mu^2\hat{u}^3\right) h_{\hat{u}}^z - \frac{E\mu\hat{u}}{k^2 f}\left(1 - \frac{k^2}{2}J\hat{u}\right)\,, \\ \hat{u}^2 f\left(\frac{h_t'^z}{\hat{u}}\right)' &= \frac{k^2}{4}\left(1 + 2J\mu^2\hat{u}^3\right) h_t^z + \mu\hat{u}^2\left(1 - \frac{k^2}{2}J\hat{u}\right) f a_z'\,, \\ 0 &= \left[\left(1 - \frac{k^2}{2}J\hat{u}\right)\left(f a_z' - \mu h_t^z\right)\right]'\,, \end{aligned}$$

and they can be solved following [44]. Let us sketch the reasoning as follows.

The quantity we are interested in is the electric DC conductivity. Considering the perturbations, the current reads

$$J_z = 2\lim_{\hat{u}\to 0} f a_z'\,.$$

Therefore, the conductivity can be calculated as $\sigma_{DC} = J_z/E$ if we obtain an expression for $f a_z'$ at the boundary. The third equation of motion is a total derivative and it can therefore be directly integrated. In particular, it can be integrated between the boundary and the horizon, giving an expression for one half of the current

$$f a_z'\Big|_{\hat{u}\to 0} = \left(1 - \frac{k^2}{2}J\right)\left(f a_z' - \mu h_t^z\right)\Big|_{\hat{u}\to 1}\,, \tag{4.3}$$

where we have used that $h_t^z = 0$ at the boundary.

Imposing regularity of the metric at the horizon helps us relate the different metric perturbations as

$$2 f h_{\hat{u}}^z\Big|_{\hat{u}\to 1} = -h_t^z\Big|_{\hat{u}\to 1}\,. \tag{4.4}$$

However, the first of the three equations of motion is actually an algebraic equation for $h^z_{\hat{u}}$, so we can find an expression for $2fh^z_{\hat{u}}$ that holds for any $\hat{u}$. It reads

$$2fh^z_{\hat{u}} = \frac{E\mu\hat{u}}{k^2}\frac{2-k^2J\hat{u}}{1+2J\mu^2\hat{u}^3}\,. \tag{4.5}$$

So far we have obtained the current in terms of horizon quantities in (4.3) and we know how to relate the two metric perturbations around the horizon (4.4). The missing piece is obtaining a'_z near the horizon. However, if we take the second of the equations of motion around the horizon and substitute h^z_t according to (4.4) and (4.5), we can solve for a'_z near the horizon in the resulting algebraic equation and obtain the remaining piece. The DC conductivity finally reads

$$\sigma_{\mathrm{DC}} = \left(1-\frac{k^2}{2}J\right)\left[1+\frac{\left(1-\frac{k^2}{2}J\right)4\mu^2}{k^2\left(1+2J\mu^2\right)}\right]\,.$$

This expression is the dimensionless conductivity where we have set the horizon to $\hat{u}_h = 1$. It is qualitatively of the same form as the one discussed in the AdS$_4$ model in [59]. In particular, it also vanishes for $k^2J = 2$ and it can even become negative in some range of the parameters, thus indicating an instability. We also note that for $J = 0$ the dimensionless conductivity obeys a similar bound as the one proven for holographic matter in 2+1 dimensions in [61]. However, since our main interest is the anomalous transport coefficients, we do not further investigate the properties of σ_{DC}.

4.3.2 Chiral magnetic conductivity

We now introduce the magnetic field as a perturbation by

$$\begin{aligned}
A_y &= \epsilon Bx\,,\\
A_z &= \epsilon a_z(\hat{u})\,,\\
g_{tz} &= \frac{\epsilon}{\hat{u}}h^z_t(\hat{u})\,.
\end{aligned}$$

The equations of motion for the perturbations are

$$\begin{aligned}
-4\kappa B\mu &= \left[\left(1-\frac{k^2}{2}J\hat{u}\right)\left(fa'_z-\mu h^z_t\right)\right]'\,,\\
\hat{u}^2 f\left(\frac{h'^z_t}{\hat{u}}\right)' &= \frac{k^2}{4}\left(1+2J\mu^2\hat{u}^3\right)h^z_t+\mu\hat{u}^2\left(1-\frac{k^2}{2}J\hat{u}\right)fa'_z\\
&\quad -2B\lambda f\left(3k^2\hat{u}+16\mu^2u^2+12f-12\right)\,.
\end{aligned}$$

The strategy to integrate these equations is as follows. First we solve the equation for a_z by writing

$$h^z_t = \frac{f}{\mu}a'_z+\frac{4\kappa B\hat{u}+c_1}{1-\frac{k^2}{2}J\hat{u}}\,,$$

where c_1 is an integration constant. Imposing regularity, the presence of a mass term for the graviton fluctuations fixes h_t^z to zero in the horizon, so $c_1 = -4\kappa B$. However, if $k^2 = 0$ this is not the case and c_1 cannot be fixed through this argument. It eventually corresponds to the choice of frame in a hydrodynamic setup.

With this expression for h_t^z, the equation corresponding to this field is now converted into a third order equation for a_z. It can be integrated and the reamaining three integration constants can be fixed by demanding regularity of the solutions on the horizon. Without going into the details of the solution, we note that this procedure results in the asymptotic expansions

$$a_z = 4\kappa B\mu\hat{u} + O(\hat{u}^2)\,,$$
$$h_t^z = -\left(\kappa\mu^2 + 8\lambda\pi^2T^2 - \lambda\frac{k^2}{2}\right)B\hat{u}^2 + O(\hat{u}^3)\,.$$

The result is independent of J, and k only appears on the metric perturbations. If we apply the usual holographic dictionary the energy-momentum tensor is given by $T_{0z} = 4g'_{tz}(\hat{u} = 0)$ [70], which coincides with our t_{0z}^{EH}, and therefore k would also appear on the transport coefficient. However, we have already argued that we need also to include $(t_\lambda)^{ij}$ and $u^{ik}K_{kj}$ to compute the correct energy current. Their contributions read

$$(t_\lambda)^{ij} = 0\,,$$
$$(u\cdot K)^{ij} = 2k^2B\lambda\delta^{i(0}\delta^{z)j}\,.$$

Using this, we can relate the components Θ^{i0} of the covariant membrane stress tensor to the energy current and finally obtain

$$\vec{J} = 8\kappa\mu\vec{B}\,,$$
$$\vec{J}_E = (4\kappa\mu^2 + 32\lambda\pi^2T^2)\vec{B}\,.$$

Taking into account (4.2), these are the usual expressions for the chiral magnetic effect in the covariant charge and energy currents. Please notice the results here have the opposite sign from those in Section 2.2.3. We have taken the opposite sign convention for the magnetic field.

4.3.3 Chiral vortical conductivity

Vorticity is introduced in this language as a gravitomagnetic field B_g in the z direction. The relation between vorticity and the gravitomagnetic field follows from observing that $\omega^i = \frac{1}{2}\epsilon^{ijk}\partial_j u_k$. In the rest frame in which $u^\mu = (1,\vec{0})$, the gravitomagnetic vector potential is the mixed space-time component of the metric in the dual field theory $ds^2 = -dt^2 + 2\vec{A}_g\cdot d\vec{x}dt + d\vec{x}^2$. The relation is subsequently fixed to be $\vec{B}_g = -2\vec{\omega}$ if we define the gravitomagnetic field as $\vec{B}_g = \vec{\nabla}\times\vec{A}_g$.

We now take the perturbations to be

$$A_y = \epsilon B_g \hat{u} \mu x \,,$$
$$A_z = \epsilon a_z(\hat{u}) \,,$$
$$g_{ty} = \frac{\epsilon f(\hat{u})}{\hat{u}} B_g x \,,$$
$$g_{tz} = \frac{\epsilon}{\hat{u}} h_t^z(\hat{u}) \,,$$

where we have already considered, from the intuition gained in Section 3.4, that the gravitomagnetic field produces a current due to drag when the chemical potential is present.

The equations of motion for the perturbations are

$$\begin{aligned} 4\kappa B_g \mu^2 \hat{u} + B_g \lambda \left(20 f'^2 - 16 \left(f - 1 \right) \frac{f'}{\hat{u}} + \frac{16\mu^2}{3} \hat{u}^2 f' \right) \\ = \left[\left(1 - \frac{k^2}{2} J \hat{u} \right) \left(f a_z' - \mu h_t^z \right) \right]' \,, \end{aligned}$$
$$\begin{aligned} \hat{u}^2 f \left(\frac{h_t'^z}{\hat{u}} \right)' = & \frac{k^2}{4} \left(1 + 2J\mu^2 \hat{u}^3 \right) h_t^z + \mu \hat{u}^2 \left(1 - \frac{k^2}{2} J \hat{u} \right) f a_z' \\ & - \lambda B_g \mu f \left(17k^2 \hat{u} + (172/3)\mu^2 \hat{u}^3 + 80 f - 80 \right) \,. \end{aligned}$$

The strategy to integrate these equations is the same as before. We first solve the equations for a_z by writing

$$\begin{aligned} h_t^z = & \frac{f}{\mu} a_z' + 32 B_g \frac{1 - \hat{u}}{1 - \frac{k^2}{2} J \hat{u}} \Big((-16\hat{u}^2 - 16\hat{u} - 16 - 144\hat{u}^4 + 48\hat{u}^3)\lambda\mu^3 - 72\kappa\mu \left(1 + \hat{u} \right) \\ & + \left(576\hat{u}^3 - 24k^2\hat{u} - 144k^2\hat{u}^3 + 192\hat{u}^2 - 24k^2 + 192\hat{u} + 24k^2\hat{u}^2 + 192\lambda \right) \mu \\ & + \frac{\lambda}{\mu} (-576\hat{u} - 36k^4\hat{u}^2 + 144k^2 - 576\hat{u}^2 + 288k^2\hat{u}^2 - 576 - 9k^4 + 144k^2\hat{u}) \Big) \,, \end{aligned}$$

where we have explicitly substituted the blackening factors and chosen the integration constant such that h_t^z vanishes at the horizon. The vanishing of the metric perturbation is again a consequence of imposing regularity, unless $k^2 = 0$. The equation for h_t^z is now converted into a third order equation for $a_z(u)$ and it can be solved with the same boundary conditions as in the case of the magnetic field. The details of the lengthy solutions are not interesting to us. The resulting asymptotic expansions are

$$a_z = (2\kappa\mu^2 + 16\pi^2 T^2 \lambda) B_g \hat{u} + O(\hat{u}^2) \,,$$
$$h_t^z = - \left(\frac{2}{3} \kappa \mu^3 + 16\lambda\mu\pi^2 T^2 \right) B_g \hat{u}^2 + O(\hat{u}^3) \,.$$

In contrast to the case with magnetic field, the asymptotic expansions are completely independent of the disorder parameters J and k, even for the energy current. This fits our expectations, because the new contributions that have to be taken into account in the membrane stress tensor depend only on the external electromagnetic fields but not on the external gravitomagnetic fields. The final result for the response due to gravitomagnetic field reads

$$\vec{J} = \left(4\kappa\mu^2 + 32\pi^2 T^2 \lambda\right) \vec{B}_g \,,$$
$$\vec{J}_E = \left(\frac{8}{3}\kappa\mu^3 + 64\lambda\mu\pi^2 T^2\right) \vec{B}_g \,.$$

Remembering that gravitomagnetic field and vorticity are related by $\vec{B}_g = -2\vec{\omega}$, these are the usual responses of a chiral fluid due to vorticity.

4.4 Discussion

Chiral magnetic and chiral vortical effects are anomaly induced dissipationless transport phenomena. In this chapter we have shown that in a simple holographic model of disorder their expression does not change. This result could have been expected and, moreover, the response in the charge current could have been inferred from [60]. However, the result of the energy current has required the involved construction of membrane currents from the previous chapter. It therefore serves as a further consistency check of the construction and a practical example where the use of such generalization of the energy-momentum tensor is required.

The mixed gauge gravitational Chern-Simons term modifies the definition of the holographic energy-momentum tensor because of its higher derivative nature. In our construction, we allow the extrinsic curvature to vary independently of the metric and then we demand that the energy-momentum tensor satisfies a constraint equation in which the extrinsic curvature acts like an external source conjugate to the operator u^{ij}. Although our background is asymptotically AdS, even to first order in the gravitational Chern-Simons coupling λ the new terms in the energy-momentum tensor give a nonvanishing contribution. This contribution appears due to the fact that the asymptotic expansion of the metric contains a constant term, i.e. $g_{tt} = -\frac{1}{\hat{u}} + \frac{k^2}{4} + O(\hat{u})$ and $g_{\hat{u}\hat{u}} = \frac{1}{4\hat{u}} + \frac{k^2}{16} + O(\hat{u})$. This constant term is a direct consequence of the presence of a background of massless scalar fields. Furthermore, the contributions of t_λ^{ij} and $(u \cdot K)^{ij}$ are crucial to restore the usual form of the chiral magnetic effect in the energy current.

Let us make some further technical remarks regarding the details of the computation before moving on to the next chapter. First of all, we have seen that in the theory with a massive graviton, regularity at the horizon imposes the vanishing of one of the integration constants that are usually free for translation symmetric theories, thus choosing a frame in the hydrodynamic sense. In particular, it fixes a frame that should be interpreted as the disorder rest frame [123, 132].

In addition, in our gravity action we have included two terms that involve higher derivatives: the gravitational Chern-Simons term and the charge disorder term proportional to the coupling J. Due to their higher derivative nature, they should be understood from an effective field theory point of view as perturbative couplings that are somehow subleading in the large N expansion.

This is for granted in the Chern-Simons contributions, as seen in the previous chapter, because we only consider linear response in the magnetic field and the vorticity. However, it is not so for the J coupling. In fact, we adopt the approach of [14, 59] of taking such coupling in a non-perturbative way in order to obtain comparable results. The interesting results are precisely the existence of an unphysical region in parameter space in which the DC conductivity becomes negative and the fact that the anomalous transport coefficients are independent of J and can be computed for any value.

Finally, let us make a connection to the results in [90], in which the Stückelberg field appears in the gauge sector, instead of associated to gravity, and it thus breaks gauge invariance in the bulk. In that case the chiral magnetic effect changes from its usual form, contrary to what happens here. Our understanding of this difference is that massive gravity breaks spatial translation invariance but there is no anomalous conductivity associated to conserved momentum. The anomalous terms in Θ^{0i} must be interpreted as an energy current and not as the (non-conserved) momentum density. In this picture, the currents corresponding to momentum conservation are then the purely spatial components, in the same way that the energy and charge currents are associated to energy and charge conservation, even in the presence of anomalies. It would be very interesting then to engineer a mechanism in holography that breaks energy conservation.

Chapter 5

Holographic Weyl semimetals

The holographic Weyl semimetal is a model for strongly coupled topological semimetals. The model presents a topological quantum phase transition separating a topological phase with non-vanishing anomalous Hall conductivity from a trivial state. In this chapter we investigate how this phase transition depends on the parameters of the scalar potential i.e. the mass and the quartic self coupling. Our results suggest that the quantum phase transition persists for a large region in parameter space. We then compute the axial Hall conductivity which, from the algebraic structure of the axial anomaly, one would expect to be $1/3$ of the electric Hall conductivity. We find that this holds once a non-trivial renormalization effect on the external axial gauge fields is taken into account.

This chapter is mostly based on [37]. The original manuscript appeared in a time when there were some concerns in the community about the correctness of using holographic nonsupersymmetric bottom-up models, as these theories might possess unstable vacuum solutions [120]. Thus, we included a discussion in Section 4 of [37] on how the results of the quantum phase transition could be reproduced in a well-known top-down model. It is based on a consistent truncation of type IIB supergravity and has been used before in the study of holographic superconductors [10]. However, we have decided not to include the discussion about the axial Hall effect in this top-down model in the thesis, since it does not provide new insight.

The chapter is organized as follows: in Section 5.1 we discuss the motivations to propose and analyze a holographic model of Weyl semimetals; in Section 5.2, we present the model and the different operators; in Section 5.3, we use the existence of a critical gravitational solution as a guiding principle to explore the universality of the quantum phase transition and its dependence on the free parameters of the model; in Section 5.4, we review the first results concerning the model and include our results for the axial magnetic effect; and, finally, in Section 5.5 we conclude the chapter with a summary of the project.

5.1 Motivation

A Weyl semimetal is a topological state of matter whose electronic quasiparticles are chiral fermions [149, 142]. It presents exotic transport properties, like anomalous Hall effect, that can be understood as effects due to chiral anomalies [102]. The band structure of Weyl semimetals is characterized by pointlike singularities in the Brillouin zone, around which electronic excitations can be understood as left- or

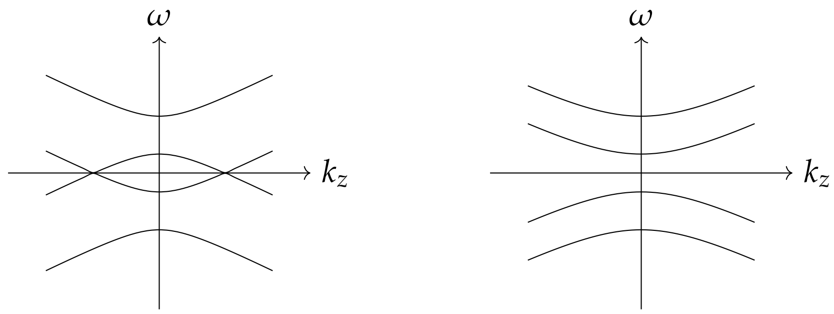

FIGURE 5.1: Spectrum of the Weyl semimetal model. In the left panel ($b^2 > M^2$), the topological phase is depicted, with the two Weyl nodes in the spectrum separated by a distance $2\sqrt{b^2 - M^2}$ in momentum space. In the right panel ($b^2 < M^2$), the system is gapped with a gap given by $2\sqrt{M^2 - b^2}$ and therefore it represents the trivial phase.

right-handed Weyl spinors. These quasiparticles always appear in pairs of opposite handedness, according to the Nielsen-Ninomiya theorem and the spinors from one of those pairs can be separated in momentum space when time-reversal symmetry is broken. When trying to explain the anomalous Hall effect in a manner compatible with Fermi liquid theory, such that all the physics was explained via quasiparticles with energies around the Fermi level, it was realized that the Weyl spinors could be understood as monopoles of the Berry curvature in momentum space [69]. The charge of this monopole is a topological invariant and therefore it is still relevant when interactions are included [151].

There exists a quantum field theoretical model whose behavior around the band touching points describes accurately a Weyl semimetal [35]. It consists on a Dirac fermion charged under an external gauge field and with a Lorentz breaking coupling $\vec{b}$. The Lagrangian reads

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu + M\right)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi - \bar{\psi}\gamma_5 \vec{\gamma}\cdot\vec{b}\psi \,. \tag{5.1}$$

The spectrum of this theory can be easily computed by Fourier transforming the resulting equations of motion and expressing the frequencies for which the determinant of the differential operator vanishes as a function of momentum. It can be seen in Figure 5.1 that there are two very different regimes depending on the relative value of $|\vec{b}|$ and $|M|$. For simplicity and without loss of generality, in the rest of the chapter we take the Lorentz breaking coupling $\vec{b}$ along the z-direction, i.e. $\vec{b} = b\vec{e}_z$.

If $|b| > |M|$, the spectrum is ungapped and the wave function around the crossing points is well-described by a Weyl fermion. The separation of the Weyl cones is $2(b^2 - M^2)^{1/2}$, so the system can be described by an effective Lagrangian like (5.1) with $M = 0$ and $b_z = (b^2 - M^2)^{1/2}$. On the contrary, the system is gapped for $|b| < |M|$ and the effective Lagrangian is that of a massive Dirac fermion with $M = (M^2 - b^2)^{1/2}$. Therefore, this model presents two phases, the topologically nontrivial Weyl semimetal phase and a trivial insulating phase, which are separated by a quantum phase transition. However, in both phases the same symmetries are broken by a nonzero value of M and b. Therefore, it is a topological phase transition and the Landau classification for phase transitions does not apply. In addition, in the presence of additional massless Dirac fermions the trivial phase is not gapped and then it represents a trivial semimetal instead of an insulator. In fact, in the holographic model that we will present below, the nontopological phase is a semimetal. The order parameter of the transition is the anomalous Hall effect

$$\vec{J} = \frac{1}{2\pi^2}\vec{b} \times \vec{E}\,, \tag{5.2}$$

which is zero for the insulating phase and nonzero for the topological phase, and the responsible for its appearance is the axial anomaly (2.5).

One might wonder if it is possible to construct a model at strong coupling that reproduces the relevant physical phenomena present in Weyl semimetals i.e if it is possible to propose a model that possesses a quantum phase transition between a phase with anomalous Hall effect and another one without it. The reason why this is a sensible concern is that the Fermi velocity in a Weyl semimetal is low compared to the speed of light and this can be related to an effective large fine structure constant similarly to graphene. This line of reasoning motivated the work to propose holographic models of Weyl semimetals. In this chapter we review the model presented in [103] and later expanded in [105, 104], but there have been some other approaches in which the holographic fermion spectral functions were studied [85, 86, 73, 66].

Holography has proved itself useful for the understanding of strongly correlated relativistic systems, including superconductors [71], strange metals [112, 40] or lattice systems [78]. It has also been crucial for a deeper understanding of anomalous transport, as extensively discussed in this thesis, so it is very natural to use the intuition gained from those works in order to study Weyl semimetals. Furthermore, in holography some of the subtleties in the computation of the anomalous Hall effect are not problematic. In field theory, the anomalous Hall effect appears as a one-loop contribution to the polarization tensor and some ambiguities arise related to its regularization [84]. They can be resolved either by matching to a tight-binding model [62, 144] or by introducing chiral edge states at the boundaries, called Fermi arcs, to cancel the anomaly [58]. In holography, however, demanding gauge invariance solves all those problems and the nontrivial renormalization is mapped to the dynamics in the radial coordinate. In this chapter such a nontrivial renormalization will be very relevant in the computation of the axial magnetic effect.

5.2 The holographic model

The action of the model is

$$\begin{aligned} S = \int d^5x\sqrt{-g}\Bigg[&\frac{1}{16\pi G}\left(R+\frac{12}{L^2}\right) - \frac{1}{4}F^2 - \frac{1}{4}F_5^2 - (D_\mu\Phi)^* D^\mu\Phi - V(\Phi) \\ &+ \frac{\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau}A_\mu\left(F^5_{\nu\rho}F^5_{\sigma\tau} + 3F_{\nu\rho}F_{\sigma\tau}\right)\Bigg] + S_{GH}, \end{aligned} \tag{5.3}$$

where G is Newton's constant, L is the AdS length, κ is the Chern-Simons coupling constant, we have already fixed the cosmological constant $\Lambda = -6/L^2$ that gives

AdS spacetime in 5 dimensions and S_{GH} is the Gibbons-Hawking term given by (3.11). From now on we set $16\pi G = L = 1$.

The field content of the action is motivated from the expected symmetries of the dual field theory. We introduce a vector gauge field V_μ and an axial gauge field A_μ, whose field strengths are respectively $F = dV$ and $F_5 = dA$, in order to account for the electromagnetic and axial $U(1)$ symmetries of the system. The scalar field Φ is only charged with respect to the axial field via the covariant derivative $D_\mu = \partial_\mu - iqA_\mu$ and has a quartic potential $V(\Phi) = m^2|\Phi|^2 + \frac{\lambda}{2}|\Phi|^4$. Explicit breaking of the axial $U(1)$ symmetry can be achieved by switching on the non-normalizable mode. We will later relate precisely this non-normalizable mode to a mass term in the dual theory. The final ingredient in the action required to make the axial symmetry anomalous is the five-dimensional Chern-Simons term with the particular choice for the coefficients shown above. They are chosen such that the gauge variation of the action mimics the VVA and AAA anomalies of Dirac fermions and preserves the vector-like gauge symmetry.

Variation of the action gives the following equations of motion

$$
\begin{aligned}
0 =& G_{\mu\nu} - 6g_{\mu\nu} + \frac{1}{8}g_{\mu\nu}F^2 - \frac{1}{2}F_\mu{}^\rho F_{\nu\rho} + \frac{1}{8}g_{\mu\nu}F_5^2 - \frac{1}{2}F^5_\mu{}^\rho F^5_{\nu\rho} \\
&+ \frac{1}{2}g_{\mu\nu}\left[(D_\rho\Phi)^* D^\rho\Phi + m^2\Phi^*\Phi + \frac{\lambda}{2}(\Phi^*\Phi)^2\right] - (D_{(\mu}\Phi)^* D_{\nu)}\Phi\,, \\
0 =& \nabla_\nu F^{\nu\mu} + 2\kappa\epsilon^{\mu\nu\rho\sigma\tau}F^5_{\nu\rho}F_{\sigma\tau}\,, && (5.4) \\
0 =& \nabla_\nu F_5^{\nu\mu} + \kappa\epsilon^{\mu\nu\rho\sigma\tau}\left(F^5_{\nu\rho}F^5_{\sigma\tau} + F_{\nu\rho}F_{\sigma\tau}\right) + iq\left(-\Phi^* D^\mu\Phi + (D^\mu\Phi)^*\Phi\right)\,, && (5.5) \\
0 =& \nabla_\mu(D^\mu\Phi) - iqA^\mu D_\mu\Phi - m^2\Phi - \lambda(\Phi^*\Phi)\Phi,
\end{aligned}
$$

where ∇_μ is the gravitational covariant derivative.

We can define the consistent currents $\mathcal{J}^a$ and $\mathcal{J}_5^a$ as the variation of the on-shell action with respect to the associated gauge fields

$$
\mathcal{J}^\mu = \lim_{r\to\infty}\frac{\delta S}{\delta V_\mu} = \lim_{r\to\infty}\sqrt{-g}\left(F^{\mu r} + 4\kappa\epsilon^{r\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right)\,, \tag{5.6}
$$

$$
\mathcal{J}_5^\mu = \lim_{r\to\infty}\frac{\delta S}{\delta A_\mu} = \lim_{r\to\infty}\sqrt{-g}\left(F_5^{\mu r} + \frac{4\kappa}{3}\epsilon^{r\mu\nu\rho\sigma}A_\nu F^5_{\rho\sigma}\right)\,. \tag{5.7}
$$

As already commented on, these expressions might look different from the ones written above, but they give equivalent results. The difference is due to the use of the Poincaré radial coordinate r, instead of the Fefferman-Graham radial coordinate $\hat{r}$, and the inclusion of the square root of the metric determinant. The square root of the bulk metric determinant here accounts in the language of Chapter 3 for the normal vector and the square root of the induced metric determinant that appears on the integrals over the constant $\hat{r}$ hypersurfaces Σ. Some of the arguments in this chapter are more clearly understood in the context of the holographic RG, so we will later make use of r-dependent membrane currents. For the sake of clarity, we denote them in this chapter as $\mathcal{J}^\mu_{(m)}$ and $\mathcal{J}^\mu_{5(m)}$, and they can be obtained from (5.6) and (5.7) by dropping the limit to the boundary.

Finally, the Ward identities can be directly found by the use of the radial component of (5.4) and (5.5), and they read

$$\begin{aligned}
&\partial_a \mathcal{J}^a = 0\,,\\
&\partial_a \mathcal{J}^a_5 = -\frac{\kappa}{3}\epsilon^{abcd}\left(F_{ab}F_{cd} + 3F^5_{ab}F^5_{cd}\right) + iq\sqrt{-g}\left(\Phi^* D^r\Phi - (D^r\Phi)^*\Phi\right)\,.
\end{aligned} \tag{5.8}$$

Again, we could do as in Chapter 4 and match the anomaly coefficients to the field theory ones.

Before entering into the details of the calculations, let us characterize the physical meaning from the dual theory's point of view of the Lagrangian parameter space:

- The **bulk mass** m^2 determines the scaling dimension of the operator dual to Φ, according to:
$$\Delta_\Phi = \frac{d + \sqrt{d^2 + 4m^2}}{2}\,,$$
where $d = 4$ for our case. The most natural choice, and the one made in the first works on the model [105], is $m^2 = -3$. This choice gives $\Delta_\Phi = 3$. Therefore, the operator dual to Φ has the dimension of a fermion bilinear mass term in four dimensions and its source has dimension one, so it can be taken as a boundary mass M. It is the most natural choice indeed because it gives the right interpretation of the last term in (5.8) as the mass term in the Adler-Bell-Jackiw anomaly (2.5). If $\Delta_\Phi \neq 3$, however, the mass has to be defined by taking appropriate powers of the non-normalizable mode, as discussed below when we introduce the holographic dictionary (5.11).

- The **quartic coupling** λ is a measure of the effective number of degrees of freedom that are not decoupled in the infrared for the trivial phase. This can be understood in terms of the holographic relationship between the rank of the gauge group and the cosmological constant, as explained below in Section 5.3. In particular, at vanishing quartic coupling the theory loses all degrees of freedom in the IR and becomes strongly coupled, while for vanishing gravitational coupling the charged degrees of freedom are negligible.

- Finally, the **charge** q modulates the mixing between the operators dual to Φ and A_M. In the limit $q \to 0$ these two operators do not mix along the RG, and all the interesting physics is lost. Applying the techniques from Chapter 3, we can better understand these mixing, and also the difference to the vector current [105]. Let us express the equations of motion (5.4) and (5.5) in terms of the membrane currents
$$\begin{aligned}
&\frac{d}{dr}\left(\mathcal{J}^\mu_{(m)} + \sqrt{-g}4\kappa\epsilon^{r\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right) = 0\,,\\
&\frac{d}{dr}\left(\mathcal{J}^\mu_{5(m)} + \sqrt{-g}\frac{2}{3}\kappa\epsilon^{r\mu\nu\rho\sigma}A_\nu F^5_{\rho\sigma}\right) = 2\sqrt{-g}q^2\phi^2\left(A^\mu - \frac{1}{q}\partial^\mu\theta\right)\,,
\end{aligned} \tag{5.9}$$
where we have decomposed the complex scalar according to $\Phi = \phi e^{i\theta}$ and assumed no dependence on the boundary coordinates. Understanding it as an RG equation, it is apparent how the properties of the axial coupling will match

between the UV and the IR only in the limit $q \to 0$. On the contrary, the first equation is the total radial derivative of the vector current independently of the point in parameter space and it lets us express the one-point function of the current completely in terms of horizon data.

We are looking for solutions which are asymptotically AdS. Our background must present anisotropy due to a nonzero axial field. This axial field will be later related to the separation of the nodes b in the field theory model (5.1). Therefore, our zero temperature ansatz is

$$T = 0: \quad ds^2 = u\left(-dt^2 + dx^2 + dy^2\right) + \frac{dr^2}{u} + hdz^2\,, \quad \Phi = \phi\,, \quad A = A_z dz\,,$$

where we have implicitly assumed that the separation between the nodes is along the z-axis, without loss of generality.

We expect our holographic model of Weyl semimetals to reproduce at zero temperature the topological quantum phase transition from the field theory model. Thus, it must interpolate between a topological phase with a nonzero Hall conductivity and a trivial phase in which the Hall conductivity vanishes. Indeed, when one tries to solve the equations of motion asymptotically near the horizon for zero temperature using this ansatz, one sees that the system has three kinds of solutions. These turn out to correspond to the trivial phase, the non-trivial phase and the critical point located between the other two phases. In particular, if we now plug this background in the RG equation for the vector membrane current (5.9), one can see that the transport coefficient of the anomalous Hall effect (5.2) for this action gives

$$\sigma^H_{xy} = 8\kappa A_z(0)\,. \tag{5.10}$$

Thus, a solution with $A_z(0) \neq 0$ will be topological while the trivial phase will appear when A_z vanishes at the horizon. We include this result here to motivate the search for the three solutions that give rise to the critical point and the two phases, but we have not been very rigorous. All the details are covered in Section 5.4.

For finite temperature, we expect to find an AdS black hole geometry, so our finite temperature ansatz is

$$T \neq 0: \quad ds^2 = -udt^2 + f\left(dx^2 + dy^2\right) + \frac{dr^2}{u} + hdz^2\,, \quad \Phi = \phi\,, \quad A = A_z dz\,.$$

In this case there is only one solution at the horizon. It fits our expectation, because at finite temperature the phase transition becomes a smooth crossover.

The phase structure is labeled in the field theory model by the ratio M/b, which is the only physically meaningful dimensionless parameter at zero temperature due to conformal symmetry. In general, if the critical value is $(M/b)_c$, then for $(M/b) < (M/b)_c$ we have a topologically nontrivial solution and for $(M/b) > (M/b)_c$ we have a trivial solution, as shown in Figure 5.3. When $(M/b)_c$ goes to infinity, it means that the trivial phase has shrunk to only one point. The mass M, which explicitly breaks the axial $U(1)$ symmetry, can be introduced by demanding that the non-normalizable mode of the scalar does not vanish. The distance between the nodes b, on the other hand, can be simply represented by a constant axial gauge

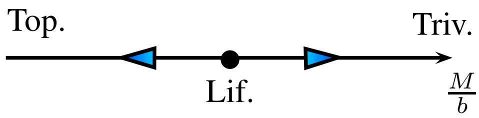

FIGURE 5.2: Schematics of the RG flow of the model. At low energy an unstable critical point is present at a certain value of M/b, corresponding to a Lifshitz scaling solution. Small deviations in M/b around this point will make the system flow in the infrared to either the topological phase to the left or to the trivial one to the right.

field. Thus we impose the following boundary conditions

$$\lim_{r\to\infty} r\Phi^{\gamma} = M\,, \qquad \lim_{r\to\infty} A_z = b\,, \tag{5.11}$$

such that the ratio M/b is dimensionless independently of Δ_Φ the dimension of the operator dual to Φ. In particular, we have introduced $\gamma = (4-\Delta_\Phi)^{-1}$ to account for the dependence of Δ_Φ on the bulk mass m^2.

5.3 Universality of the topological quantum phase transition

Our aim for this section is to determine the universality of the topological quantum phase transition. We assume that the existence of a critical solution monitors the appearance of a quantum phase transition. Therefore, we try to understand how the change in the parameters of the model affects the possibility to find a critical solution and the value of $(M/b)_c$. The first step to map (m^2, q, λ) into the critical value $(M/b)_c$ is to find an asymptotic solution in the IR that we can integrate to the boundary and match to the boundary conditions given in (5.11).

The equations of motion for the zero temperature background read

$$\begin{aligned}
0 =& A_z'' + \left(\frac{2u'}{u} - \frac{h'}{2h}\right) A_z' - \frac{2q^2\phi^2}{u} A_z\,,\\
0 =& \phi'' + \left(\frac{h'}{2h} + \frac{2u'}{u}\right)\phi' - \left(\frac{q^2A_z^2}{h} + m^2 + \lambda\phi^2\right)\frac{\phi}{u}\,,\\
0 =& \frac{3u''}{4u} + \frac{3u'^2}{8u^2} + \frac{1}{4}\phi'^2 - \frac{3}{u} + \frac{\phi^2}{4u}\left(m^2 + \frac{\lambda}{2}\phi^2 - \frac{q^2A_z^2}{h}\right) - \frac{1}{8h}A_z'^2\,,\\
0 =& u'' - \frac{u'h'}{2h} + \frac{2}{3}u\phi'^2 - \frac{2}{3h}q^2A_z^2\phi^2\,,
\end{aligned}$$

where primes denote, as in the rest of the chapter, derivatives with respect to Poincaré's radial coordinate r. Following [105], we expect three solutions. One of them is a scaling solution and it actually satisfies these equations of motion all along the bulk: it is the critical solution. The other two, however, are only asymptotic solutions in the IR and they represent the two different phases. Although we are mainly interested in the critical solution as a signal of existence for the phase transition, let us also discuss the other two.

Asymptotic solutions for topological and trivial phase

The approach to solve the equations asymptotically begins by taking an ansatz with the expected asymptotics

$$u = u_0 r^2\,, \quad h = h_0 r^2\,, \quad A_z = A_0\,, \quad \phi = \phi_0\,.$$

If we solve for the accompanying coefficients, it can be seen that this particular ansatz gives two solutions. Either ϕ or A_z has to vanish at the horizon and for each case the coefficient of u acquires different values ($u_0 = 1$ and $u_0 = 1 + m^4/(24\lambda)$, respectively). For the solution with vanishing A_0, ϕ_0 is also fixed to $\sqrt{-m^2/\lambda}$. In contrast, the coefficient on h is undetermined for both cases, thus allowing for a rescaling such that $h = r^2$.

According to (5.10), the case with $A_0(0) = 0$ is the trivial solution and, therefore, the case with $\phi_0(0) = 0$ represents the topological solution. In order to find the full asymptotic solutions, understanding the computation in a perturbative way makes the computation simpler. We then take irrelevant perturbations (u_1, h_1, ϕ_1 and A_1) around these two solutions, that must be regular at the horizon and subleading for small r. We expand the equations to first order in perturbations and keep only the leading terms. It can be seen that on both cases the equations for u_1 and h_1 decouple and the solutions of those equations have to be taken to zero from regularity. However, the equations for ϕ_1 and A_1 look rather different in the two cases.

On one hand, the leading terms in the equations for the **topological phase** read

$$\begin{aligned} 0 &= \phi_1'' + \frac{5}{r}\phi_1 - \frac{q^2 A_0^2}{r^4}\phi_1\,, \\ 0 &= A_1'' + \frac{3}{r}A_1' - \frac{2q^2 A_0}{r^2}\phi_1\,. \end{aligned}$$

We first solve for ϕ_1 and then we plug in the solution on the other equation. The final result gives

$$\begin{aligned} u &= r^2\,, \\ h &= r^2\,, \\ \phi &= \frac{\phi_1}{(2qA_0)^{5/2}}\frac{e^{-\frac{qA_0}{r}}}{r^{3/2}}\,, \\ A_z &= A_0 + \frac{\phi_1^2}{64q^5A_0^6}e^{-\frac{2qA_0}{r}}r\,. \end{aligned}$$

On the other hand, the leading terms in the equations for the **trivial phase** read

$$\begin{aligned} 0 &= \phi_1'' + \frac{5}{r}\phi_1' + \frac{48\lambda m^2}{(24\lambda + m^4)\,r^2}\phi_1\,, \\ 0 &= A_1'' + \frac{3}{r}A_1' + \frac{2q^2m^2}{\lambda\left(1 + \frac{m^4}{24\lambda}\right)r^2}A_1\,. \end{aligned}$$

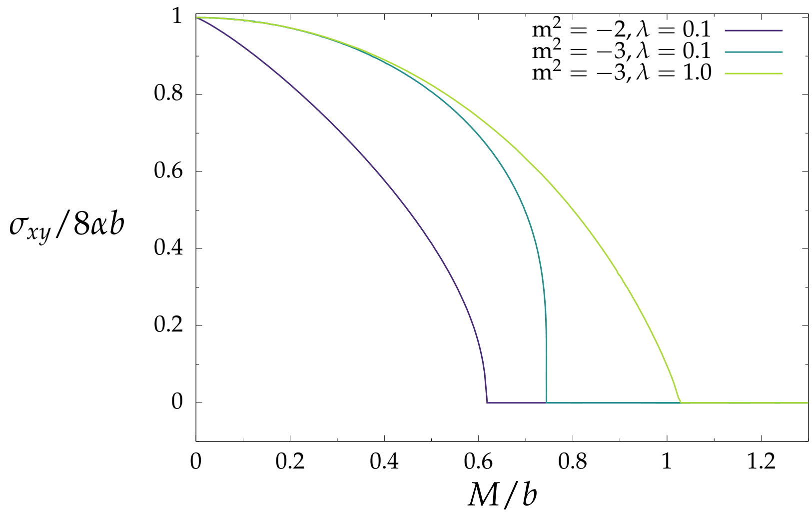

FIGURE 5.3: The zero temperature Hall conductivity for different values of the model's parameters. It can be observed how the critical value for the M/b parameter changes in the different cases.

We can solve the A_1 equation and then the ϕ_1 equation. The complete solution is

$$\begin{aligned} u &= \left(1 + \frac{m^4}{24\lambda}\right) r^2 \,, \\ h &= r^2 \,, \\ A_z &= A_1 r^{\beta_1} \,, \\ \phi &= \sqrt{\frac{-m^2}{\lambda}} + \phi_1 r^{\beta_2} \,, \end{aligned}$$

where

$$\begin{aligned} \beta_1 &= -1 + \sqrt{1 - \frac{48 q^2 m^2}{m^4 + 24\lambda}} \,, \\ \beta_2 &= -2 + 2\sqrt{1 - \frac{12\lambda m^2}{m^4 + 24\lambda}} \,. \end{aligned}$$

Let us now move on to the scaling solution that will be associated to the critical point.

Critical solution

It is possible to obtain a Lifshitz-like geometry with a non-trivial scaling exponent in the z-direction that is solution of the equations of motion for the whole bulk. However, we expect to maintain AdS asymptotics in the boundary, such that the dual field theory is Lorentz invariant. Therefore, we understand this scaling solution also as IR asymptotics that will have to be numerically integrated to the conformal boundary. This geometry represents the **critical solution** and we will focus on it for the rest of this section. In particular, we will concentrate on the computation of

$(M/b)_c$ for each choice of the parameters in the Lagrangian. In general, there is only one critical solution for a given value of m^2, λ and q.

We begin with a scaling ansatz

$$u = u_0 r^{2\alpha}\,, \quad h = h_0 r^{2\beta}\,, \quad A_z = A_0 r^{\gamma}\,, \quad \phi = \phi_0 r^{\delta}\,.$$

Imposing the equations of motion, we find

$$u = u_0 r^2, \quad h = h_1 r^{2\beta}, \quad A_z = r^{\beta}, \quad \phi = \phi_0,$$

where h_1 is h_0/A_0^2. The four parameters (u_0, h_1, β and ϕ_0) are functions of (m^2, q and λ), according to

$$\begin{aligned}
0 &= 3h_1(u_0-1) - \frac{1}{8}u_0\beta^2 + \frac{1}{4}\phi_0^2(h_1 m^2 - q^2) + \frac{1}{8}\phi_0^4 h_1\lambda,\\
0 &= 2u_0 h_1(1-\beta) - \frac{2}{3}q^2\phi_0^2,\\
0 &= 3u_0\beta - 2q^2\phi_0^2,\\
0 &= m^2 h_1 + q^2 + \lambda h_1\phi_0^2.
\end{aligned}$$

Solving the last three equations for u_0, h_1 and β, we obtain the following relations:

$$u_0 = \frac{2q^2\phi_0^2}{3\beta}\,,$$

$$h_1 = -\frac{q^2}{m^2+\lambda\phi_0^2}\,, \tag{5.12}$$

$$\beta = -\frac{2q^2}{m^2+\lambda\phi_0^2-2q^2}\,. \tag{5.13}$$

The first equation turns into a third order equation in ϕ_0^2. Not all the solutions of the third order equation are physical. Therefore, it is necessary to impose some constraints, which can be summed up in the following way:

Regularity: β needs to be larger than zero.

Reality: A_z and ϕ need to be real, because we want to recover a real value of M and b in the UV. This gives us the conditions that β is real and ϕ_0^2 is real and positive.

Null-energy condition: We impose the null-energy condition $T^{MN}\xi_M\xi_N \geq 0$, with ξ_M any future-pointing light-like vector field. Taking $\xi = \frac{1}{\sqrt{u}}dt + \sqrt{h}dz$ and making use of Einstein's equations gives $T^{MN}\xi_M\xi_N = G^{MN}\xi_M\xi_N = 1-\beta$, so the null-energy condition reads $\beta \leq 1$. This condition also enforces realness of the axial field since it ensures the positivity of h_1, as it becomes obvious from (5.12) and (5.13) that $\beta = 2/(2+h_1^{-1})$.

Once the physical solution of the system has been found, one can flow to asymptotic AdS in the UV by perturbing the system with irrelevant perturbations around the Lifshitz fixed point

$$\begin{aligned} u &= u_0 r^2 (1 + \delta u \, r^\chi), \\ h &= h_1 r^{2\beta} (1 + \delta h \, r^\chi), \\ A_z &= r^\beta (1 + \delta a \, r^\chi), \\ \phi &= \phi_0 (1 + \delta\phi \, r^\chi). \end{aligned}$$

The fact that all the perturbations have the same scaling exponent χ follows immediately from the linearized form of the equations of motion. Solving these again for the new parameters (χ, δu, δh, δa, $\delta\phi$) it is important to enforce realness and regularity on this solutions and this requires χ to be real and positive. The equation has seven solutions but there is only one solution satisfying these two conditions. The other four parameters can be expressed as a function of only one of them. The possibility to integrate numerically the solution to the UV using the equations of motion will only depend on the sign of this free parameter. Finally, the boundary conditions (5.11) allows us to complete the map from (m^2, q, λ) to $(M/b)_c$ and study the appearance and location of the phase transition.

Decoupling of degrees of freedom

So far we have concentrated on using the possibility to find a scaling solution as criterion for the existence of a quantum phase transition. However, we can use the holographic dictionary to find a more physical argument to justify our results. From (2.20) and (2.21), we can relate the number of degrees of freedom N to the Anti de Sitter length L and the string length l_s as

$$N \propto \left(\frac{L}{l_s}\right)^4 . \tag{5.14}$$

The UV/IR correspondence of holography maps the geometry near the boundary of spacetime to the ultraviolet conformal fixed point of the dual QFT, while the deep bulk geometry (which in the finite temperature case is a black hole's horizon) contains information about the infrared degrees of freedom. If both the ultraviolet and infrared geometries are asymptotically AdS then (5.14) implies that

$$\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}} = \left(\frac{L_{\mathrm{IR}}}{L_{\mathrm{UV}}}\right)^4 ,$$

where $N_{\mathrm{UV/IR}}$ denote the UV/IR degrees of freedom and $L_{\mathrm{UV/IR}}$ represents the UV and IR Anti de Sitter length scales. Of course, in order for classical gravity to be a valid description, both N_{UV} and N_{IR} are formally infinite, but their ratio remains a finite quantity. L_{IR} is defined implicitly through the infrared cosmological constant $\Lambda_{\mathrm{IR}} = -\frac{12}{L_{\mathrm{IR}}^2}$ whose value can be computed explicitly as $\Lambda_{\mathrm{IR}} = -\frac{12}{L_{\mathrm{UV}}^2} + V(\Phi_{\mathrm{IR}})$. Then, the counting of degrees of freedom can be expressed as

$$\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}} = \left(\frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}\right)^2 . \tag{5.15}$$

Since AdS is an Einstein manifold, the ratio between the IR and UV degrees of freedom can also be expressed in terms of the scalar curvature

$$\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}} = \left(\frac{R_{\mathrm{UV}}}{R_{\mathrm{IR}}}\right)^2 . \tag{5.16}$$

Please notice that this quantity can never be greater than one as we flow between two AdS spacetimes, since it would violate the holographic version of the celebrated C-theorem [54, 52].

While this reasoning applies in a straightforward way in the case of AdS asymptotics, it has to be generalized if the IR geometry is not AdS. We use the scalar curvature as an estimator of the degrees of freedom, since it is the only constant length scale which is physically observable. In our holographic model there are three different possible infrared geometries which all have different curvatures. Using (5.15) and (5.16), a straightforward computation gives

$$\begin{aligned}
\left(\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}}\right)_{\mathrm{top}} &= 1, \\
\left(\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}}\right)_{\mathrm{triv}} &= \frac{1}{\left(1+\frac{(m^2)^2}{24\lambda}\right)^2}, \\
\left(\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}}\right)_{\mathrm{c}} &= \left(\frac{10}{u_0(6+\beta(3+\beta))}\right)^2 .
\end{aligned}$$

Notice that only the result for the critical solution depends on the charge q. Consistency with our weak coupling intuition would require that the decoupling of degrees of freedom is intermediate between the two phases, according to

$$\left(\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}}\right)_{\mathrm{top}} > \left(\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}}\right)_{\mathrm{c}} > \left(\frac{N_{\mathrm{IR}}}{N_{\mathrm{UV}}}\right)_{\mathrm{triv}} .$$

Results

The result of the two methods described above are presented in Figures 5.4 and 5.5, where we study dependence on λ and m^2, respectively. In the case of the dependence on λ we clearly see that, for large enough λ, the critical M/b parameter diverges. This implies, as the topological phase still exists for lower M/b than the critical one, that the transition to the trivial semimetal phase ceases to take place, making the system impossible to be gapped at low energies. It is interesting that the limit $\lambda \to 0$ reaches smoothly a finite constant value of M/b. However, this is not easy to analyze, as the gravitational backreaction on the geometry has to be arbitrarily big in order to compensate for the scalar field's energy density, which scales as λ^{-1}.

The argument using the infrared degrees of freedom strengthens our conclusions. In fact, it can be seen that, as we increase λ, the infrared degrees of freedom of the critical solution become eventually less than those of the trivial one. In this case the phase transition ceases to take place, as the gapped phase would have too

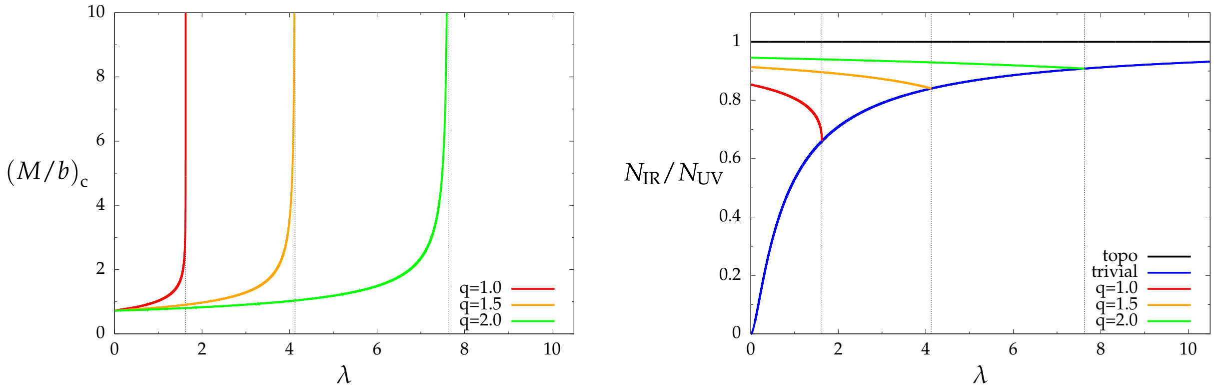

FIGURE 5.4: Phase transition as a function of the model's parameters: (left) critical M/b as a function of λ, (right) infrared degrees of freedom as a function of λ. Divergences in the M/b values and crossing in the counting of degrees of freedom signal the impossibility for the phase transition to take place. They are emphasized by a dotted vertical line and coincide between the two methods.

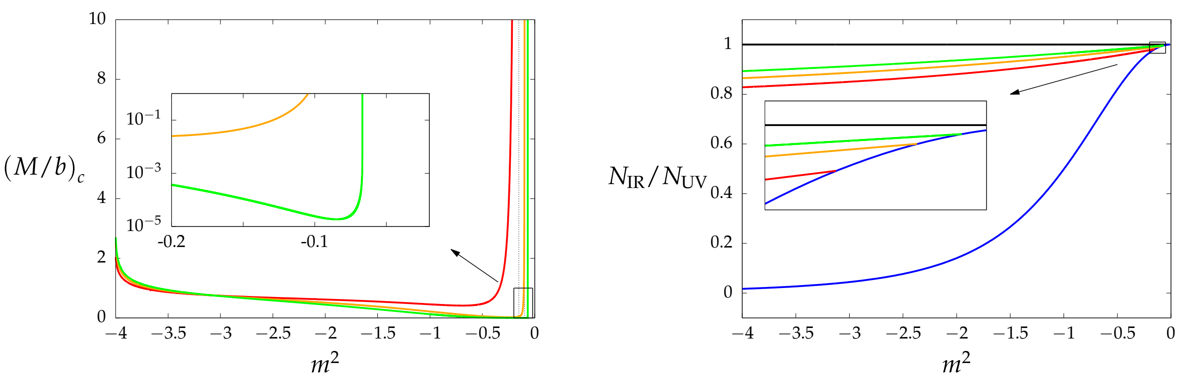

FIGURE 5.5: Phase transition as a function of the model's parameters: (left) critical dimensionless ratio M^γ/b ($\gamma^{-1} = 4 - \Delta_\Phi$) as a function of m^2, (right) infrared degrees of freedom as a function of m^2. Points at which the transition ceases to take place coincide. To avoid cluttering, we include the legend here: $q = 1.0$ (red), $q = 1.25$ (orange) and $q = 1.5$ (green). As above, the black line in the right panel represents the topological phase and the blue line represents the trivial phase.

many degrees of freedom to be reached from the critical point and we would be left with a system that only presents a topological phase. Furthermore, the limiting λ values coincide between the two computations within numerical precision.

Looking now at Figure 5.5, we notice two things. First of all, we get a similar steep curve to the one in the left panel of Figure 5.4 when we approach a critical lower value for m^2 that is closer to zero as we increase the charge. This again signals the disappearance of a trivial phase. However, the limit $m^2 = 0$ is never reached. In such a limit, the dimensionless coupling $M = \lim_{r\to\infty}\phi$ would act as an effective mass for the gauge field A_M through the quadratic gauge coupling, giving $m_A = q^2M^2$. This behavior receives the name of holographic Stückelberg mechanism [97, 32, 68, 90]. The other interesting behavior, which can be seen in the inset of Figure 5.5 (A), is that the lower bound $(M/b)_c > 0$ does not get crossed in any case. Thus, the topological solution always exists.

This is confirmed by the counting of degrees of freedom, which crosses the ones from the trivial phase at the same values of m^2 where $(M/b)_c$ diverges. If, on the other hand we had lost the topological phase, we would have seen a crossing of the horizontal line $N_{\mathrm{IR}}/N_{\mathrm{UV}} = 1$. Interestingly, the limit $m^2 \to -4$ does not present any particular problem. This conclusion is however not complete, as we describe only dual operators with dimensions $2 \leq \Delta_\Phi \leq 4$. Operators of dimensions $1 \leq \Delta_\Phi \leq 2$ can be described using the alternative quantization scheme for big enough negative masses ($-4 \leq m^2 \leq -3$ for a scalar field in five dimensions). In this case we would identify the leading contribution at the conformal boundary as the one point function of the operator dual to Φ instead of its coupling.

As a final remark we notice that the system seems to display an odd behavior in the case in which $q^2 < 1$. Two critical solutions with different infrared degrees of freedom can be found. However, the system admits no solution in between the two critical solutions, as can be seen both from numerics and from the counting of degrees of freedom. This unphysical situation may be the consequence of some fundamental bound on gravitational systems, with profound consequences on the allowed dual operator's charges. We leave this problem for future studies.

5.4 Axial conductivity and infrared screening

The holographic model of Section 5.2 has a nonzero transversal conductivity in the topologically nontrivial phase

$$\mathcal{J}^i = \epsilon^{ijk}\sigma_j^H E_k\,, \tag{5.17}$$

where σ_i^H is the electric anomalous Hall conductivity. From a quantum field theoretical point this is a consequence of the anomaly giving rise to a coupling in the effective action of the form

$$W[V,A,b] = W[V,A,0] + \frac{N_A^f N_c^2}{24\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}b_\mu\left(3V_\nu F_{\rho\sigma} + A_\nu F^5_{\rho\sigma}\right)\,, \tag{5.18}$$

where the parameter b_μ is the effective infrared value of the axial gauge field [62]. In holography this result can be extracted in a rather straightforward way from the five dimensional Maxwell-Chern-Simons equations of motion (5.4), as we have already advanced in (5.9). When expressed in terms of the membrane current $J^a_{(m)}$, the μ components of (5.4) read

$$\frac{d}{dr}J^a_{(m)} + \sqrt{-g}\nabla_b F^{ba} + 8\kappa\sqrt{-g}\epsilon^{abcrd}\partial_b\left(A_c F_{rd}\right) = 0\,.$$

If one turns on a boundary electric field with no time or spatial dimension, we can find the conservation of the current zero mode. Upon radial integration between the boundary and the horizons, the one-point function of the current can be expressed in terms of the value at the horizon of the membrane current, which represents the low energy degrees of freedom of the theory. It can be obtained from the Bianchi identity $\epsilon^{\mu\nu\rho\sigma\tau}\partial_\rho F_{\sigma\tau} = 0$, taking μ, ν to be two of the spatial directions, that the electric field does not get renormalized

$$\frac{d}{dr}\tilde{F}_{0i} = \frac{d}{dr}\tilde{E}_i = 0\,.$$

Therefore, as we also used in Section 3.4, the $F^{r\mu}$ term in the membrane current can be expressed at the horizon as a function of the electric field through the in-falling boundary conditions, following [82]. This term gives rise to a longitudinal conductivity, which vanishes at zero temperature and will be omitted for simplicity. Finally, explicit evaluation of the Chern-Simons part of (5.6) on the horizon gives, for the transversal part

$$\left\langle \mathcal{J}^i \right\rangle = 8\kappa\epsilon^{ijk}A_j(r_H)\tilde{E}_k.$$

We can identify $A_j(r_H)$ with the effective infrared coupling b_i and obtain the result in agreement with (5.17).

In the following we focus on computing the Hall conductivity for the anomalous axial current $\mathcal{J}^\mu_5$. The calculation in this case has to be done resorting to numerical methods, because of the coupling to the scalar field ϕ. However, as we will see towards the end of the section, we would expect the structure of the anomalies to completely determine the form of such a coefficient because of the relation (5.18). This is, in fact, possible. Properly taking into account the RG flow to low energies allows us to successfully match the holographic computation to the weak coupling results. These axial transport properties are of high physical interest, as they can appear in condensed matter studies. Axial vector fields arise at an effective level due to lattice strain [39, 122, 63]. This computation also describes a topological signature of anomalous $U(1)^3$ theories in the presence of charged matter. This can then be used to directly detect the family of phase transitions presented in Section 5.3 even if only the anomalous symmetry is present.

5.4.1 Axial Hall conductivity

The axial Hall conductivity cannot be computed easily by solving the RG flow of the current because of the symmetry breaking of the scalar field. We therefore resort to the Kubo formulae introduced in Section 2.3. In what follows, σ_{jk} is related to the anomalous Hall conductivity through $\sigma^H_i = \frac{1}{2}\epsilon_i{}^{jk}\sigma_{jk}$. The expressions for the conductivities are

$$\sigma_{ik} = \lim_{\omega\to 0} \frac{1}{i\omega} \langle \mathcal{J}_i \mathcal{J}_k \rangle \; (\omega, \vec{k} = 0),$$
$$\sigma^5_{ik} = \lim_{\omega\to 0} \frac{1}{i\omega} \langle \mathcal{J}^5_i \mathcal{J}^5_k \rangle \; (\omega, \vec{k} = 0).$$

In holography one can obtain the retarded Green's function by studying the fluctuations around the background for the gauge fields dual to the currents and imposing infalling boundary conditions. The Hall conductivity is the off-diagonal part of the above equation so, for the vector conductivity, we need to turn on fluctuations on the x- and y-directions of the vector gauge field i.e. $\delta V_{x/y} = v_{x/y}(r)e^{-i\omega t}$. The equations of motion for these fluctuations, both for finite and zero temperature, are

$$v''_x + \left(\frac{h'}{2h} + \frac{u'}{u}\right) v'_x + \frac{\omega^2}{u^2} v_x + \frac{8i\omega\kappa A z'}{u\sqrt{h}} v_y = 0\,,$$
$$v''_y + \left(\frac{h'}{2h} + \frac{u'}{u}\right) v'_y + \frac{\omega^2}{u^2} v_y - \frac{8i\omega\kappa A z'}{u\sqrt{h}} v_x = 0\,.$$

On the contrary, the axial gauge field fluctuations ($\delta A_{x/y} = a_{x/y}(r)e^{-i\omega t}$) will also produce fluctuations on the metric ($\delta g_{xz} = g_{xx}\, h^x{}_z e^{-i\omega t}$ and $\delta g_{yz} = g_{yy}\, h^y{}_z e^{-i\omega t}$). For finite temperature, the complete set of coupled equations of motion reads

$$\begin{aligned}
0 =& a''_x + \left(\frac{h'}{2h} + \frac{u'}{u}\right) a'_x + \frac{\omega^2}{u^2} a_x + \frac{8i\omega\kappa A'_z}{u\sqrt{h}} a_y - \frac{2q^2\phi^2}{u} a_x - \frac{f A'_z}{h} h^{x\prime}{}_z\,,\\
0 =& a''_y + \left(\frac{h'}{2h} + \frac{u'}{u}\right) a'_y + \frac{\omega^2}{u^2} a_y - \frac{8i\omega\kappa A'_z}{u\sqrt{h}} a_x - \frac{2q^2\phi^2}{u} a_y - \frac{f A'_z}{h} h^{y\prime}{}_z\,,\\
0 =& h^{x\prime\prime}{}_z + \left(\frac{2f'}{f} + \frac{u'}{u} - \frac{h'}{2h}\right) h^{x\prime}{}_z + \left(\frac{f''}{f} - \frac{u''}{u} + \frac{f'h'}{2fh} - \frac{u'h'}{2uh}\right) h^x{}_z + \frac{\omega^2}{u^2} h^x{}_z\\
&+ \frac{A'_z}{f} a'_x + \frac{2q^2 A_z \phi^2}{fu} a_x\,,\\
0 =& h^{y\prime\prime}{}_z + \left(\frac{2f'}{f} + \frac{u'}{u} - \frac{h'}{2h}\right) h^{y\prime}{}_z + \left(\frac{f''}{f} - \frac{u''}{u} + \frac{f'h'}{2fh} - \frac{u'h'}{2uh}\right) h^y{}_z + \frac{\omega^2}{u^2} h^y{}_z\\
&+ \frac{A'_z}{f} a'_y + \frac{2q^2 A_z \phi^2}{fu} a_y\,.
\end{aligned}$$

The zero temperature case follows from this one taking f to be equal to u. Besides the coupling to the gravity sector, the main difference between both sets of equations is that in the axial gauge field case we cannot take the equation of motion to be a total derivative, in the same way it already happens for the background. On the contrary, a term involving the scalar field appears, giving a non-trivial RG flow along the bulk which will take a major role in the understanding of the results obtained for the conductivity.

The coupling to the gravity sector cannot be avoided. We can make both equations for the vector field decouple from each other with a particular set of basis i.e.

$v_\pm = v_x \pm i v_y$. However, with an analogous basis one can only reduce the set of four equations of the axial field to two sets of two equations each. Therefore, the linear response coefficients relating the non-normalizable mode with the normalizable mode will have to be obtained taking care of holographic operator mixing [93]. We will try to briefly explain the method followed. From now on, we focus on the axial gauge field sector.

We start by expanding the action to second order in perturbations of the fields. Then, we Fourier transform this quantity and take only positive momenta. At this stage we multiply the fields by the proper power of r, to make their leading term in the near-boundary asymptotics constant. Finally, having the action expressed in this way, we impose the equations of motion and get the on-shell action, which will have the following form:

$$S = \int dk_> \left[2A_{IJ}\Phi^I_{-k}\Phi'^J_{-k} + B_{IJ}\Phi^I_{-k}\Phi^J_k \right]_{r_h}^{r_b} = \int dk_> \left[\varphi^I_{-k}\mathcal{F}_{IJ}(k,r)\varphi^J_k \right]_{r_h}^{r_b},$$

where Φ^I_k is the field mode associated to momentum k and φ^I_k is its value at a cut-off close to the boundary that we will call r_Λ. The remaining key ingredient is to use the bulk-to-boundary propagator (BBP) to connect the first form of the on-shell action with the second one, by expressing the bulk fields $\Phi^I_k(r)$ in terms of their value at the cut-off φ^I_k:

$$\Phi^I_k(r) = F^I{}_J(k,r)\varphi^J_k,$$
$$\Phi^I_{-k}(r) = F^I{}_J(-k,r)\varphi^J_{-k} = \varphi^J_{-k}F^{\dagger I}_J(k,r).$$

With all this, we can then take

$$\mathcal{F}(k,r) = 2F^\dagger A F' + F^\dagger B F,$$

which will give us minus the retarded Green's function in the limit where the cut-off is shifted to the boundary

$$G^R_{IJ}(k) = -\lim_{r\to\infty}\mathcal{F}_{IJ}(k,r).$$

The remaining piece of the method is the construction of the BBP. When one solves asymptotically the equations of motion near the horizon, it can be seen that each field is defined up to an unspecified integration constant. We can take these values for the different fields to form a vector and use a basis of this vector space. The normalization of this basis doesn't matter because we will end up normalizing the BBP to be the unit matrix at the cut-off. Therefore, for simplicity we will take the linearly independent combinations of the integration constants to be sets of ones with a minus one that is in a different position for each solution. For each of the elements of this basis, we will integrate numerically the fields to the UV and construct a solution matrix $H(k,r)$ where the rows represent each fluctuation and the columns represent each of the different solutions

$$H^I{}_J(k,r) = \Phi^I{}_{(J)}(k,z).$$

Finally, the BBP can be obtained as:

$$F(k,r) = H(k,r) \cdot H(k,r_\Lambda)^{-1}.$$

Now we have explained the method, we will include the field vector and the A and B matrices in the second-order on-shell action for finite temperature. Zero temperature case would be obtained substituting f by u. In our case, we don't need to take into account the power of r on the leading term of the asymptotic expansion, since $a_{x/y}$ have a constant leading term and, although the leading term of the metric perturbations is of order r^2, the variables $h^x{}_z$ and $h^y{}_z$ also have a constant leading term.

$$\Phi = \begin{pmatrix} a_x \\ h^x{}_z \\ a_y \\ h^y{}_z \end{pmatrix},$$

$$A = \begin{pmatrix} -\frac{\sqrt{h}f}{2} & 0 & 0 & 0 \\ 0 & -\frac{f^3}{4\sqrt{h}} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{h}f}{2} & 0 \\ 0 & 0 & 0 & -\frac{f^3}{4\sqrt{h}} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \frac{f^2 A_z'}{\sqrt{h}} & -\frac{8i\omega\kappa A_z}{3} & 0 \\ 0 & -\frac{3f^2 f'}{2\sqrt{h}} & 0 & 0 \\ \frac{8i\omega\kappa A_z}{3} & 0 & 0 & \frac{f^2 A_z'}{\sqrt{h}} \\ 0 & 0 & 0 & -\frac{3f^2 f'}{2\sqrt{h}} \end{pmatrix}.$$

The application of this methodology requires the construction of a solution for the background fields and the study on top of them of solutions to the equations of motion of the fluctuations. The background fields are obtained following the method explained in Section 5.3, where the IR solutions of both phases and of the critical point are integrated numerically to the UV. Then, we look for the asymptotic solutions of the fluctuations near the horizon and impose regularity and infalling boundary conditions, which could alternatively be seen as regularity on Eddington-Finkelstein coordinates. Thus, we have an analytical form for the boundary conditions at the horizon that we use for the numerical integration of the fluctuations. It depends only on the holographic coordinate r, on the frequency ω and on the integration constants to which we made reference above, which will be either ones or minus ones. The numerical solutions of the fluctuations are then plugged into the H matrices, the background fields are included in the A and B matrices and all this allows us to get the Green's function.

We can safely take b to be the relevant physical energy scale throughout our computation. Therefore the only dimensionless parameter in the problem is M/b for zero temperature and both M/b and T/b for the finite temperature case, as well as the frequency ω/b for the perturbations. When taking the small frequency limit we need to take care of the region in which we will integrate our equations of motion. In particular, we need to take the frequency to be smaller than the distance r at which we took the IR limit of the integration region. The system is also very sensitive to the maximum r at which we integrate, since the metric fluctuations give notable numerical errors for large integration regions. Therefore, since the solutions we obtain for these fields have the form of a domain wall and the final value is already reached at a quite low value of r we looked at the value of the holographic coordinate at which the boundary value of the fluctuations was saturated and take this as our UV result.

5.4.2 Current renormalization.

As shown in [58, 159, 62], the anomalous Hall effect can be seen at weak coupling to stem from an anomalous contribution in the low energy effective action given by the infrared coupling to the axial current $b^{\mathrm{IR}}_{\mu} J^{\mu}_{5}$, where we use the suffix IR to denote couplings of the effective low energy description of the system, in contrast to "bare" couplings in the UV description. In our language, this current insertion can be reexpressed as an extra contribution to the effective action given by

$$\delta W[V,A] = \frac{N_f^A N_c^2}{24\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} b^{\mathrm{IR}}_{\mu} \left(3V_\nu F_{\rho\sigma} + A_\nu F^5_{\rho\sigma} \right) ,$$

such that functional differentiation gives a closed expression for both the anomalous axial and vector conductivities in the presence of an (axial) electric field $E^i_{(5)}$:

$$\vec{\mathcal{J}} = \frac{N_f^A N_c^2}{2\pi^2} \vec{b}_{\mathrm{IR}} \times \vec{E} \,.$$
$$\vec{\mathcal{J}}_5 = \frac{N_f^A N_c^2}{6\pi^2} \vec{b}_{\mathrm{IR}} \times \vec{E}_5 \,.$$

We can read from these expressions the axial and vector transverse conductivities

$$\sigma_A = \frac{N_f^A N_c^2}{6\pi^2} b_{\mathrm{IR}}, \quad \sigma_V = \frac{N_f^A N_c^2}{2\pi^2} b_{\mathrm{IR}} \,,$$

from which we can predict that the ratio between the axial transverse conductivity σ_A and the vector one σ_V is

$$\frac{\sigma_A}{\sigma_V} = \frac{1}{3}.$$

This ratio should be fixed completely by the structure of the anomalies of the theory and not receive any further low energy corrections, even though the low energy effective action is hard to determine precisely as the system is gapless. Nevertheless, if we naively compute this quantity from holography, we find a rather different answer (see Figure 5.7 and 5.8). The reason why this happens is that the external fields in the infrared will in general couple with a different strength than they do in the

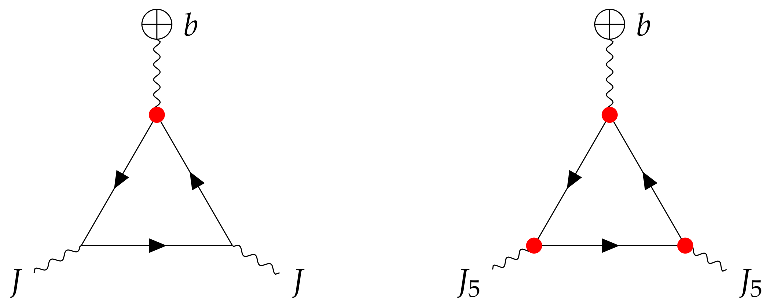

FIGURE 5.6: Diagrams corresponding to the leading corrections to the two-point functions $\langle JJ \rangle$ and $\langle J_5 J_5 \rangle$. These will be the most important contributions to vector (left) and axial (right) conductivities in the infrared effective theory. Red dots denote the renormalized coupling of the axial operators $\sim \sqrt{Z_A}$. It can be noted how the first diagram scales as $\sqrt{Z_A}$, while the other one scales as $\sqrt{Z_A^3}$. These versions of the diagrams were created using *TikZ-Feynman* [48].

ultraviolet. Thus, their infrared coupling will also appear in the DC conductivities, as they are a response of the low energy physics.

This can be thought of as a field renormalization effect due to the fact that we have sourced a charged operator in the UV theory, which is described by the scalar field Φ. It is useful to introduce a renormalization constant $\sqrt{Z_A}$ such that:

$$\sqrt{Z_A} b_\mu = b_\mu^{\mathrm{IR}}.$$

The coupling gets renormalized as we flow to the IR effective theory in the Wilsonian sense. In this case the important contributions at weak coupling come from the diagrams in Figure 5.6, where red dots account for the renormalized couplings. Thus we will take all the axial fields to be renormalized through $\sqrt{Z_A}$ in the infrared, so that the effective action reads:

$$\delta W[V,A] = \frac{N_f^A N_c^2}{24\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} b_\mu \left(3 Z_A^{1/2} V_\nu F_{\rho\sigma} + {Z_A}^{3/2} A_\nu F^5_{\rho\sigma} \right),$$

which implies a ratio between the transverse conductivities

$$\frac{\sigma_A}{\sigma_V} = \frac{1}{3} Z_A = \frac{1}{3} \left(\frac{b_{\mathrm{IR}}}{b} \right)^2.$$

In holography we have shown that the infrared effective coupling is reproduced by the horizon value of the background axial field, so that $\frac{b_{\mathrm{IR}}}{b} = \frac{A_z(0)}{b}$. We then expect:

$$\frac{\sigma_A}{\sigma_V} = \frac{1}{3} \left(\frac{A_z(r_H)}{b} \right)^2, \tag{5.19}$$

while we can recover the right $1/3$ coefficient if we express the quantity as a function of the "screened" infrared fields $A_M^{\mathrm{IR}} = \sqrt{Z_A} A_M$.

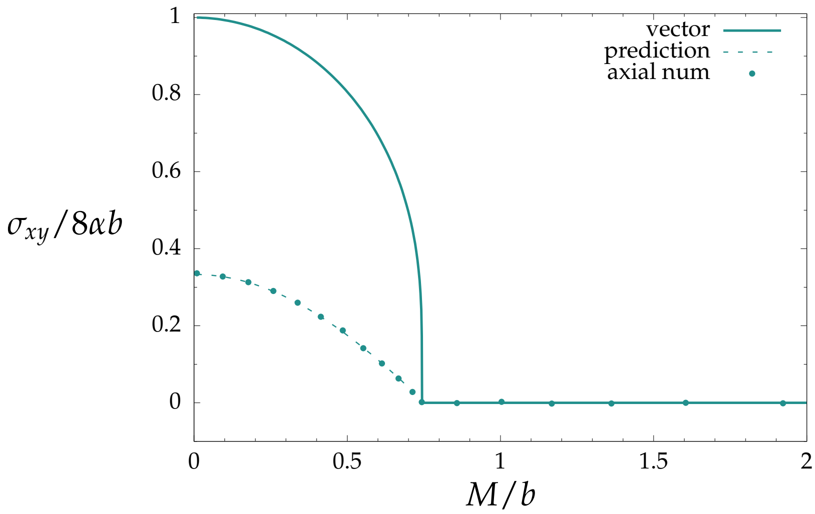

FIGURE 5.7: Normalized transversal axial conductivity for zero temperature. The line denotes the original result for the vector conductivity, dots denote numerical data for the axial conductivity and dashed lines denote the prediction (5.19).

We can now compare our prediction with the numerical results in Figures 5.7 and 5.8. We notice the remarkable fit of the prediction to the data, which still holds in the finite temperature regime, with the transition smoothed out to a crossover. Through this interpretation we can then recover holographically both the $1/3$ coefficient, coming from the UV anomaly structure of the theory, as well as the right dependence on the infrared renormalized couplings. Thus, our study displays both the high and low energy character of this phenomenon, as should be expected from its relation to the anomalies of the theory.

5.5 Discussion

In Section 5.3 we have studied the dependence of the quantum phase transition of the holographic Weyl semimetal on the parameters of the scalar potential. We have found that generically it persists but that the trivial phase becomes unaccessible for large quartic scalar self coupling or close to the marginality bound on the dual operator. From the point of view of bulk physics, those limits are related to a scalar potential that does not have a non-trivial vacuum that spontaneously breaks axial gauge symmetry.

In Section 5.4 we have computed the axial Hall conductivity and found that after taking a non-trivial renormalization of the axial gauge field into account it is precisely $1/3$ of the vector Hall conductivity. This is indeed what can be expected from general arguments based on the algebraic structure of the axial anomaly. This result is of interest for condensed matter systems. While on a fundamental level axial gauge fields are not present in nature they can appear as effects of strain in Weyl semimetals [39, 63, 122].

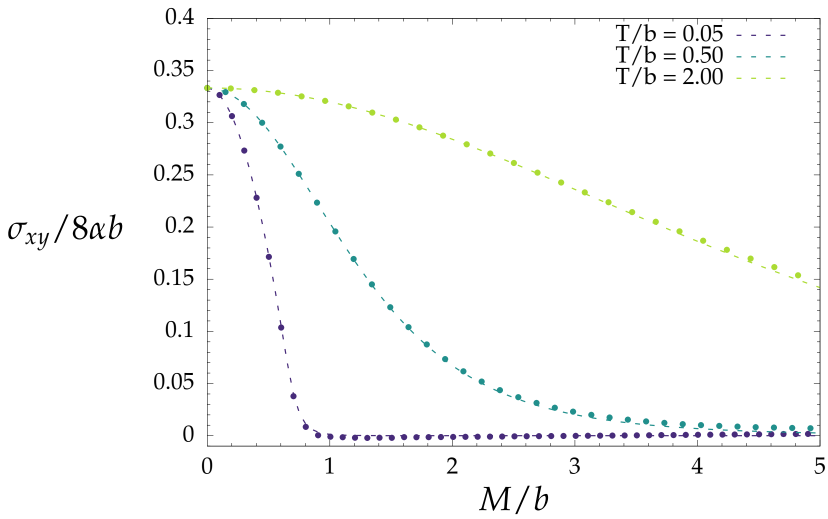

FIGURE 5.8: Normalized transversal axial conductivity for finite temperature. The computation is done for different temperatures to prove the state independence of the result.

Chapter 6

Out of equilibrium anomalous transport

We have only studied anomalous transport in equilibrium so far. However, some of the systems in which these transport phenomena are expected to play a significant role, like the quark gluon plasma produced in heavy ion collisions, are highly out of equilibrium. Therefore, it is very natural to extend our discussion on anomalous transport phenomena to out of equilibrium setups. In this chapter, we compute anomalous transport phenomena sourced by small vector and axial magnetic fields in out of equilibrium systems. We use generalized Vaidya-like metrics that include momentum relaxation to make sudden changes in the thermodynamic variables of the background and then we analyze the response induced by those changes. As a result, we find that the chiral magnetic effect shows large equilibration times that depend on the properties of the quench and we extract some general conclusions about the consequences this might have for the phenomenology of heavy ion collisions.

This chapter is based on [51] and it is organized as follows: in Section 6.1, we discuss our motivation for the study of the chiral magnetic effect out of equilibrium; in Section 6.2 we introduce the holographic model; in Section 6.3 we present the AdS Vaidya-like solutions with momentum relaxation; in Section 6.4 the linear response equations are set up; in Section 6.5 we do a preliminary analysis based on hydrodynamics which serves to establish some intuition on the out of equilibrium evolution; the core of the chapter is Section 6.6, where we compute the anomalous response far from equilibrium and we concentrate on the response of the vector current in a background of axial charge and vector magnetic field, and the response of the axial and energy currents due to axial charge and axial magnetic field; in Section 6.7 we analyze the relevant quasinormal mode spectrum and compare to the direct numerical solutions of the time evolution, and finally, in Section 6.8 we discuss the results obtained.

6.1 Motivation

An important aspect of anomalous transport is that in a subtle way it is always related to some non-equilibrium physics. At the theoretical level, the anomaly induced contribution in a gauge current like the electric current has to vanish in strict equilibrium. This result is known as Bloch theorem and it stems from the fact that the contribution to the action of a nonvanishing current that couples to a gauge field would depend on the gauge choice. In particular, a discussion in relation to the chiral magnetic effect (CME) has been given in [154]. A careful examination of the CME shows that it indeed vanishes in equilibrium due to a topological contribution from a counterterm that arises in the definition of the electric current [67]. In contrast, if a current is related to an anomalous global symmetry, it does not necessarily have a vanishing expectation value.

The reader might wonder if the motivation for studying anomalous transport out of equilibrium is only theoretical. However, the nonequilibrium behavior is also essential in experimental situations in which the CME arises, like Weyl semimetals [109] or the quark gluon plasma produced in heavy ion collisions. In Weyl semimetals out of equilibrium dynamics arise due to the application of an electric field parallel to the magnetic field, which induces the celebrated negative magnetoresistivity [129]. On the other hand, in heavy ion collisions the chiral imbalance is induced in the early far from equilibrium stages by the gluonic contribution to the axial anomaly, as discussed in [96].

So far, most of the theoretical investigations of anomaly induced transport, including the ones in the previous chapters, have concentrated on equilibrium or near-equilibrium situations which can be described by hydrodynamics [130]. However, it is clear that a much better understanding of anomaly induced transport out of equilibrium is needed. We are in the position to do it with the help of holography: besides being very useful to investigate transport in strongly coupled systems, it allows to study the out of equilibrium evolution of those systems by means of numerical relativity in AdS spaces [34].

Some of the simplest time dependent gravity solutions are Vaidya metrics, which are generated by in-falling incoherent null dust and can be obtained as analytic solutions of Einstein's equations for very simple energy-momentum and charge distributions. In this chapter, we use asymptotically anti de-Sitter charged Vaidya metrics to simulate the out of equilibrium evolution. On top of this background, we add a small magnetic field and calculate the linear response in the vector, axial and energy currents due to the axial anomaly.

The response in the energy current can be known a priori because it must be equal to the conserved momentum density. Since no additional momentum is injected into the system, its value does not change. However, this behavior can be altered if momentum ceases to be conserved. Therefore, we introduce a Vaidya-like background solution that also includes momentum relaxation through the same mechanism of Chapter 4. The existence of this generalization of Vaidya metrics has been noted before in the case of four dimensional asymptotically AdS Vaidya metrics in [152, 16]. It must be understood physically as some sort of homogeneous distribution of heavy impurities that destroy momentum.

While we emphasize that our setup represents far from equilibrium physics it is still useful to compare to hydrodynamics to gain some intuition. In general, transport can have some convective component due to the overall flow of the fluid, but this flow is impeded by the impurities and thus has to vanish when the external perturbations are switched off and the system approaches a new equilibrium. As a result, the response in the energy-momentum tensor and current is given as the sum of convective and anomaly induced parts. Momentum relaxation will destroy the convective part and only the anomalous contribution is left. The latter is dissipationless and thus it cannot be affected by the impurities, as was shown in Chapter 4 for equilibrium in holography and in [132] for an effective hydrodynamics setup. However, it is still interesting to analyze the behavior in holographic far from equilibrium evolution.

In holography out of equilibrium anomaly induced transport has been studied before [110, 8, 106]. However, all those approaches are different to ours in some aspect. For example, in [110], the authors studied a free falling charged shell of matter in AdS. While this is similar to the Vaidya approach, their computation of the CME response used a quasi-static approximation whereas we directly solve for the full time dependence. In [8], the effects of parallel magnetic and electric fields were studied, in order to simulate the negative magneto-resistivity out of equilibrium. Finally, in [106] the work revolved around the out of equilibrium behavior of the effects induced by the gravitational anomaly. Therefore, the work of this chapter is a new study that complements the already available ones.

6.2 The model

We use the following holographic action

$$S = \int d^5x\sqrt{-g}\Big[\frac{1}{16\pi G}\left(R+\frac{12}{L^2}\right)-\frac{1}{2}\left(\partial X\right)^2-\frac{1}{4}F^2-\frac{1}{4}F_5^2 \\ +\frac{\kappa}{3}\epsilon^{\mu\nu\rho\sigma\tau}A_\mu\left(3F_{\nu\rho}F_{\sigma\tau}+F^5_{\nu\rho}F^5_{\sigma\tau}\right)\Big]+S_{GH}+S_{nf}\,.$$

The model is essentially the one from (5.3), from which it inherits a vector $U(1)$ symmetry and an axial $U(1)$ symmetry which suffers from both VVA and AAA anomalies. However, it is also supplemented with the scalar kinetic term from (4.1) in order to have momentum relaxation. Besides that, we add by hand a null-fluid action S_{nf} to the model, which will allow us to obtain a Vaidya-like background solution, and the Gibbons Hawking term S_{GH} from (3.11). From now on, we fix Newton's constant G and the AdS length scale L as $16\pi G = L = 1$.

The equations of motion for the model are

$$\begin{aligned} Y^I_{(nf)} =& \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\partial^\mu X^I\right)\,,\\ J^\mu_{(nf)} =& \nabla_\nu F^{\nu\mu}+2\kappa\epsilon^{\mu\nu\rho\sigma\tau}F_{\nu\rho}F^5_{\sigma\tau}\,,\\ J^\mu_{5(nf)} =& \nabla_\nu F_5^{\nu\mu}+\kappa\epsilon^{\mu\nu\rho\sigma\tau}\left(F_{\nu\rho}F_{\sigma\tau}+F^5_{\nu\rho}F^5_{\sigma\tau}\right)\,,\\ T^{(nf)}_{\mu\nu} =& G_{\mu\nu}-\frac{6}{L^2}g_{\mu\nu}-\frac{1}{2}\partial_\mu X^I\partial_\nu X^I+\frac{1}{4}\partial_\rho X^I\partial^\rho X^I g_{\mu\nu}\\ &-\frac{1}{2}F_{\mu\rho}F_\nu{}^\rho+\frac{1}{8}F^2 g_{\mu\nu}-\frac{1}{2}F^5_{\mu\rho}F^{5\rho}_\nu+\frac{1}{8}F_5^2 g_{\mu\nu}\,, \end{aligned}$$

where the sources in the left-hand side of the equations stand for the variation of the null-fluid action with respect to the scalar, the two different gauge fields and the metric, respectively.

From the quantum field theory point of view, the definitions of the scalar operators, covariant currents and energy-momentum tensor are

$$Y^I = \lim_{\hat{r}\to\infty} \sqrt{-\gamma}\,\partial_{\hat{r}X^I}\,,$$
$$J^i = \lim_{\hat{r}\to\infty} \sqrt{-\gamma}F^{i\hat{r}}\,, \tag{6.1}$$
$$J_5^i = \lim_{\hat{r}\to\infty} \sqrt{-\gamma}F_5^{i\hat{r}}\,, \tag{6.2}$$
$$T^{ij} = \lim_{\hat{r}\to\infty} 2\sqrt{-\gamma}\left[-K^{ij} + K\gamma^{ij}\right]\,, \tag{6.3}$$

where we are implicitly using the Fefferman-Graham coordinates $ds^2 = d\hat{r}^2 + \gamma_{ij}dx^i dx^j$. Please notice that we define the currents with the appropriate induced metric, contrary to the definitions in Chapters 3 and 4. The difference stems from the fact that in those chapters the currents were defined as fields whose fluxes over the membranes gave the appropriate field theory one-point functions, while here the currents are directly defined as QFT observables. The currents satisfy the following holographic Ward identities

$$\partial_i J^i = 0\,,$$
$$\partial_i J_5^i = -\kappa\sqrt{-\gamma}\epsilon^{ijkl}\left(F_{ij}F_{kl} + \frac{1}{3}F^5_{ij}F^5_{kl}\right)\,,$$
$$\partial_i T^i{}_j = F^{ij}J_j + F_5^{ij}J_j^5 - Y^I\partial_j X_i\,.$$

6.3 Background construction

Our motivation for this work is studying the behavior out of equilibrium of anomalous transport with a small magnetic field and momentum relaxation. Therefore, our background shall represent a time-evolving homogeneous and isotropic charged state in a theory that breaks translation symmetry. The magnetic field will be later included as a perturbation on top of this background.

The simplest setup that serves our purpose consists of a black brane with time-dependent blackening factor, which in Eddington-Finkelstein coordinates has the form

$$ds^2 = -f(v,r)dv^2 + 2dvdr + r^2 d\vec{x}^2\,,$$

and a linear spatial profile for the scalars, associating each one to a certain boundary coordinate

$$X^1 = kx\,, \quad X^2 = ky\,, \quad X^3 = kz\,.$$

As the scalars couple only through derivatives, the field equations and solutions are still formally translation invariant.

We also want our background to be charged, in order to have nonzero chemical potential. As there are no magnetic or electric fields in the background, and we choose the gauge to be $V_r = A_r = 0$, the Chern-Simons terms in the equations of motion will have no contribution at the level of the background. Since these are the terms that mix both gauge fields, we can thus focus on only one of the gauge fields for the construction of the background. We stick to the vector field-strength F to avoid unnecessary cluttering of the notation, but the exact same discussion would apply to the axial gauge field sector.

If we impose that the expectation value of the charge density $\langle J^v \rangle$ is equal to the charge q, we can fix the field strength to be

$$F_{rv} = \frac{q}{r^3} \, .$$

This field strength corresponds to an external source given by

$$J^v_{(ext)} = \frac{\dot{q}}{r^3} \, ,$$

where the dot stands for derivatives with respect to v.

The blackening factor f can be obtained as solution of two different time-independent differential equations that arise as components of the Einstein's equations. One of them is second order and the other one is first order, but compatibility of both solutions imposes the extra integration constant from the second order one to vanish. We fix the other integration constant by comparing it to the standard form of the mass term in uncharged Vaidya metrics, finally giving

$$f(v,r) = r^2 \left(1 - \frac{k^2}{4r^2} - \frac{2m(v)}{r^4} + \frac{q(v)^2}{12r^6} \right) \, .$$

This corresponds to an external source given by

$$T^{(ext)}_{vv} = \frac{3\dot{m}}{r^3} - \frac{q\dot{q}}{4r^5} \, .$$

One could wonder why the mass and charge are allowed to vary with time but the momentum relaxation coefficient is not. It is important to note in this regard that their origin is very different. As it was discussed in the first paragraph of this section, momentum relaxation is a feature of the theory while the charge and the mass, necessary to have a black brane, are properties of the state. In fact, their appearance in the context of holography is very different. While the momentum relaxation coefficient is fixed when sourcing the scalars, the mass and the charge appear as integration constants in the solution of the differential equations.

The thermodynamics of the system are defined by the chemical potential and the temperature, which we compute as the difference between the boundary value and the horizon value of the zeroth component of the gauge field and the Hawking's temperature of the black brane, respectively. Their values are

$$\mu = \frac{q}{2r_H^2} \, , \qquad T = \frac{1}{4\pi} \left(\frac{k^2}{2r_H} + \frac{8m}{r_H^3} - \frac{q^2}{2r_H^5} \right) \, , \tag{6.4}$$

where r_H stands for the position of the apparent horizon. We do not consider the event horizon since its location could be changed by events that happen long after the end of our numerical simulations. Please note again that all this discussion would be the same in the presence of axial charge exchanging q by q_5, or adding both contributions if both charges were present.

One obvious concern may arise. While chemical potential and temperature cannot be defined in out of equilibrium setups, definitions (6.4) in this Vaidya background are valid at all times and react instantaneously to the changes in mass and charge. We would like to make two comments on this. On one hand, although one could feel tempted to take this expressions as true also out of equilibrium, it is important to understand that they are ill-defined in such regimes and we will only use them as a near equilibrium approximation to detect true out of equilibrium behavior. On the other hand, we want to look at the out of equilibrium behavior of anomalous transport. We use this Vaidya background in order to simplify the computations. The fact that the background seems to react to the changes instantaneously doesn't alter the fact that the computed one point functions present clear out of equilibrium behavior. We would not expect notable deviations in the qualitative picture if the procedure involved backgrounds that are generated by numerical integration, as it was done in [106].

From now on, we will change our radial coordinate to be

$$u = \frac{1}{r}\,,$$

and we denote in the rest of the chapter the derivatives with respect to u by primes. We use this coordinate in order to simplify the numerical procedure. Please do not confuse it with the coordinate $\hat{u}$ used in Chapter 4. Both radial coordinates are related according to $\hat{u} = u^2$.

6.4 Linear response computations

We are going to probe the out of equilibrium behavior of the system by switching on a small constant magnetic field, either vector or axial, and the minimal set of fluctuations required by consistency of the equations of motion at linear order in the magnetic field. In order to carefully perform the perturbative computation, we include an infinitesimal coefficient ϵ whose powers will account for the order in the expansion and we will drop all powers from the second one on.

Without loss of generality, we set our magnetic fields along the z-axis, such that $F_{xy} = \epsilon B$ or $F^5_{xy} = \epsilon B_5$ in each case. The fluctuations that will be switched on are $V_z = \epsilon V$, $A_z = \epsilon A$, $g_{vz} = \epsilon h/u^2$ and $X^3 = kz + \epsilon Z$, grouped in different subsets depending on the particular case, as will become evident below.

It is straightforward to check that the four different cases with vector or axial charge and vector or axial magnetic field reduce to two different sets of equations. The first set exists for the two cases with vector magnetic field and the second set appears for the two cases with axial magnetic field, and they read

$$B: 0 = dV' - \frac{1}{2u}dV + \frac{uf}{4}V' - 4\kappa B q_5 u^2\,, \tag{6.5}$$

$$B_5: 0 = dA' - \frac{1}{2u}dA + \frac{uf}{4}A' - 4\kappa B_5 q_5 u^2 + \frac{q_5 u}{2}H\,, \tag{6.6}$$

$$0 = dh' + \frac{5uf + u^2 f'}{2}H - q_5 u^3 dA - kdZ + k^2 h\,, \tag{6.7}$$

$$0 = dZ' - \frac{3}{2u}dZ + \frac{3uf}{4}Z' + \frac{3k}{2u}h - \frac{k}{2}H\,, \tag{6.8}$$

$$0 = \left(\frac{H}{u^3}\right)' - q_5 A' + \frac{k}{u^3}Z'\,. \tag{6.9}$$

where $H = h'$ and d stands for directional derivatives along the outgoing null geodesics

$$d = \partial_v - \frac{u^2 f}{2}\partial_u\,.$$

The other equations in each of the sets of equations are obtained by $q5 \leftrightarrow q$ and $A \leftrightarrow V$. More on this will be commented below.

Equation (6.9) involves no time derivative and thus it can be understood as a constraint that has to be fulfilled at all times. This form of (6.7) is obtained after using the constraint to exchange time derivatives by the new operator d. It is straight-forward to see that the momentum relaxation coefficient acts simultaneously as a coupling between the scalar and the metric perturbation and a mass for the metric. If there were no momentum relaxation, the scalar would decouple from the system of equations, as it will also become obvious from the quasi-normal modes. Please notice that, although not explicitly, momentum relaxation is also relevant in the equations of the gauge fields through the blackening factors.

It is mandatory to check the compatibility of the equations of motion and the constraint for the case with axial magnetic field. It can be seen that

$$-\dot{q}_5 A' = d(\text{eq.}6.9) - \partial_u\left(\frac{(\text{eq.}6.7)}{u^3}\right) - \partial_u\left(\frac{u^2 f}{2}\right)(\text{eq.}6.9) - \frac{2k}{u^3}(\text{eq.}6.8)\,.$$

Therefore, the constraint and the equations of motion are compatible only in the case when the axial charge does not vary with time. This is not a feature of momentum relaxation, since it is true even for the momentum preserving case, $k = 0$.

The definition of the directional derivatives along the outgoing geodesics gives us a time evolution equation for the different perturbations by solving for the v-derivative of each field. The use of this directional derivatives is according to the well-known method of characteristics, in which one uses information about the trajectories along which the information of the solution is transported to simplify the form of the equations to be solved. In the case with B_5, the constraint could in principle be used as the equation for h, but we got better accuracy by using the same procedure for A, h and Z and then the constraint can be used as a test that the time evolution was correct.

So far we have only discussed two of the four possible situations. The case with

q and B can be obtained from the case with q_5 and B by exchanging V by A and q_5 by q in (6.5). The case with q and B_5 can be obtained by exchanging A by V and q_5 by q in (6.6)-(6.9). Thus, the out of equilibrium behavior is grouped according to the magnetic field being vector or axial and we only need to make the computations for the two cases considered above.

However, this is rather peculiar, because according to the type of anomaly the classification is different: the case with q_5 and B_5 stems from the $U(1)_A^3$ while the other three come from the $U(1)_A U(1)_V^2$. We can better understand this by looking at the expressions these effects have in equilibrium in terms of the anomaly coefficients

$$\vec{J}_a = d_{abc}\frac{\mu_b}{4\pi^2}\vec{B}_c\,,$$
$$\vec{J}_\epsilon = d_{abc}\frac{\mu_a\mu_b}{8\pi^2}\vec{B}_c\,,$$

where $a, b, c = V, 5$ and $\vec{J}_\epsilon$ is the energy current. It can be seen, for example, for a single Dirac fermion ($d_{VV5} = d_{V5V} = d_{5VV} = d_{555} = 2$) that, while a vector magnetic field never produces an energy current unless both chemical potentials are nonzero, the axial magnetic field produces it as long as one of the chemical potentials is present.

We need to solve the two different systems numerically. In order to do that, we need boundary conditions, which we obtain from the asymptotic solutions near the boundary of AdS, and also initial conditions, which we obtain from the fact that the system is in equilibrium at the beginning.

We used a fourth order Runge-Kutta method to solve the equations both in the time and radial directions. The radial derivatives of the fields were approximated using finite differences. In particular, we used fourth order centered finite differences, except in the first two points near the boundary and the last two points beyond the horizon, where we used second order finite differences, in the appropriate combination of centered, forward or backward.

We used an initial spatial grid of 1000 (vector current) or 10000 (axial and energy current) equally distributed points between $u = 0$ and $u = 1$. This grid is supplemented with 10 extra points inside the horizon. Along the time evolution the spatial integration range is cut at each step to 10 points inside the apparent black hole horizon. The time step was taken to be at least one order of magnitude smaller than the radial step in order to guarantee stability of the algorithm.

We have seen that the computations were very sensitive to the boundary data and treating the first points right was the most important part of the numerical procedure. This is to be expected since the boundary of AdS is a regular singular point. In order to circumvent the associated complications we explicitly imposed in our equations the limit when u goes to zero, using in some cases L'Hôpital's rule with the asymptotic expansion, as we will see below.

We didn't want the operators to be sourced, so we fixed the non-normalizable modes of all the different fields to zero. The first terms of the rest of the expansion, from the normalizable mode on, read

$$\begin{aligned}
V =& V_2 u^2 + \dot{V}_2 u^3 + \mathcal{O}(u^4)\,, \\
A =& A_2 u^2 + \dot{A}_2 u^3 + \mathcal{O}(u^4)\,, \\
h =& h_4 u^4 - \frac{4kZ_4}{5} u^5 + \mathcal{O}(u^6)\,, \\
Z =& Z_4 u^4 + \frac{5\dot{Z}_4 - kh_4}{5} u^5 + \mathcal{O}(u^6)\,,
\end{aligned}$$

where all the coefficients were functions of time only and h_4 had to satisfy

$$\dot{h_4} = -kZ_4\,.$$

This constraint is clearly related to the conservation of T^{0i}, which is broken for $k \neq 0$.

We can now substitute these asymptotic solutions in the definitions of the currents ((6.1), (6.2) and (6.3)) to obtain the responses parallel to the magnetic fields, which read

$$\langle J^z \rangle = 2V_2\,, \tag{6.10}$$
$$\langle J_5^z \rangle = 2A_2\,, \tag{6.11}$$
$$\langle T^{0z} \rangle = 4h_4\,. \tag{6.12}$$

Therefore, obtaining our results boils down to extracting the normalizable modes. In order to do that we perform a least squares fit according to the series expansion and read from the normalizable modes of the fields the one-point functions of the gauge and energy currents. We also use the vanishing of the coefficients of previous powers as a check for the accuracy of the method.

As we advanced above, we also used these series expansions in the equations of motion, substituting in V, dV, A, dA, h, dh, Z and dZ, to obtain the form of the equations in the first point of the grid. After applying L'Hôpital's rule to take the limit $u \to 0$, it can be seen that the radial equations reduce at that point to

$$\begin{aligned}
&\lim_{u\to 0}(\text{eq.}6.5) = 0 = dV' + V_2\,, \\
&\lim_{u\to 0}(\text{eq.}6.6) = 0 = dA' + A_2\,, \\
&\lim_{u\to 0}(\text{eq.}6.7) = 0 = dh'\,, \\
&\lim_{u\to 0}(\text{eq.}6.8) = 0 = dZ'\,.
\end{aligned}$$

In order to proceed with the integration algorithm, we used in these equations the V_2 and A_2 that had been obtained in the fit from the previous time step. For the rest of the points in the grid, we didn't need to use any information from the fit, only the numerical results of the fields and their derivatives.

Finally, in order to obtain the initial conditions, we assumed that the system started in equilibrium and obtained the equilibrium solutions for the appropriate values of m, q_5 and k. These equilibrium solutions could be found analytically solving the static version of the equations of motion, where all the time derivatives were dropped. Their expressions are very lengthy and cumbersome. Since we don't consider them to be particularly informative, we have decided not to include them here. Our late time state will also be an equilibrium configuration and it can be checked that our late time solutions approach the appropriate solutions of the static equations.

6.5 Comparing to hydrodynamics

In order to detect true out of equilibrium phenomena, we can compare our results to near equilibrium ones obtained from hydrodynamics. We start by writing down the constitutive relations for the different currents

$$
\begin{aligned}
J^\mu =& \rho u^\mu + \sigma B^\mu + \sigma_5 B_5^\mu\,, \\
J_5^\mu =& \rho_5 u^\mu + \tilde{\sigma} B^\mu + \tilde{\sigma}_5 B_5^\mu\,, \qquad (6.13)\\
T^{\mu\nu} =& (\epsilon + p) u^\mu u^\nu + p\eta^{\mu\nu} + \xi u^{(\mu} B^{\nu)} + \xi_5 u^{(\mu} B_5^{\nu)}\,, \qquad (6.14)
\end{aligned}
$$

where parentheses on the indexes stand for symmetrization $A_{(\mu}B_{\nu)} = A_\mu B_\nu + A_\nu B_\mu$. Please note that the standard symmetrization also involves a combinatorial factor in the denominator, but we don't include it because our purpose is just obtaining more compact expressions.

Typically, relativistic hydrodynamics has an ambiguity on the choice of frame, which in holography presents itself as a choice of boundary conditions. In the cases with axial magnetic field and momentum relaxing parameter k different from zero, regularity conditions at the horizon impose one more condition on the independent modes of the series expansion: the metric perturbation has to be zero at the horizon, exactly as we already saw in Chapter 4. Therefore, it can be said that momentum relaxation chooses a preferred frame and that this frame is the disorder rest frame (the "no-drag" frame of [132]). For convenience, we also impose this condition in the rest of the cases, where we have freedom to choose the frame.

Once we choose the frame, we stick to it for the whole computation. In and near equilibrium, the transport coefficients in this frame are given by

$$
\begin{aligned}
\sigma &= 8\kappa\mu_5\,, & \sigma_5 &= 8\kappa\mu\,, \\
\tilde{\sigma} &= 8\kappa\mu\,, & \tilde{\sigma}_5 &= 8\kappa\mu_5\,, \\
\xi &= 4\kappa\mu\mu_5\,, & \xi_5 &= 4\kappa\mu_5^2\,.
\end{aligned}
$$

Any deviation from these expressions in the one-point functions of the quantum currents, (6.10), (6.11) and (6.12), will be taken as a sign of out of equilibrium physics. Please notice the sign of the transport coefficients here is the opposite to the one from Section 2.2.3. We have taken the opposite sign convention for the magnetic field.

In the cases with axial magnetic field we will be looking both at the axial current and the energy current. In relativistic hydrodynamics, the energy current T^{0i} is equal to the momentum density $P^i = T^{i0}$. The latter is usually a conserved quantity except in the case with momentum relaxation. Therefore, the response in the energy current is bound to be trivial without momentum relaxation. The chemical potentials change of course but the different anomaly induced energy current is compensated by a convective part due to flow in (6.14). This convective component is subject to dissipation in the case with momentum relaxation. We can not compute the exact value of the fluid velocity, because we only have access to the one point functions of the different currents but not to the anomalous and convective contributions separately.

However, we can compute what the flow would be near equilibrium from the point of view of hydrodynamics with momentum conservation. Since the one point function of the $0i$-components of the energy momentum tensor does not change, it always has its initial value. This means, according to the constitutive relation in (6.14) that

$$\langle T^{0z}\rangle = 4\kappa(\mu_5^{in})^2 B_5 = (\epsilon+p)v_z + 4\kappa\mu_5^2 B_5\,,$$

where the superscript *in* means it is the initial value of the axial chemical potential. We can solve for the fluid velocity

$$v_z = \frac{4\kappa B_5}{\epsilon+p}\left[(\mu_5^{in})^2 - \mu_5^2\right]\,.$$

Once we have computed the fluid velocity, we can use the constitutive relation of the axial current (6.13) to see what the value of the one point function will be according to hydrodynamics. It gives

$$\langle J_5^z\rangle = \rho_5 v_z + 8\kappa\mu_5 B_5 = 8\kappa\mu_5 B_5 + \frac{4\kappa B_5\rho_5}{\epsilon+p}\left[(\mu_5^{in})^2 - \mu_5^2\right]\,.$$

At this point we use again the constitutive relations and the holographic dictionary with the background metric and gauge field to obtain the transport coefficients

$$\begin{aligned}\epsilon &= \langle T^{00}\rangle = 6m\,,\\ p &= \langle T^{ii}\rangle = 2m\,,\\ \rho_5 &= \langle J^0\rangle = q_5\,,\end{aligned}$$

and substitute to obtain the equilibrium value of the axial current considering flow with momentum conservation

$$\langle J_5^z\rangle = 8\kappa\mu_5 B_5 + \frac{\kappa B_5 q_5}{2m}\left[(\mu_5^{in})^2 - \mu_5^2\right]\,.$$

We use these hydrodynamic expressions for the currents as benchmarks for near equilibrium evolution in the following (see Figures 6.2, 6.4 and 6.6) where μ_5 is defined as in (6.4).

6.6 Results

In all the cases studied, we performed a quench in the mass of the form

$$m = m_0 + \frac{m_f - m_0}{2}\left(1 + \tanh\left(\frac{v}{\tau}\right)\right)\,. \tag{6.15}$$

We chose the masses in order to fix initial and final horizon positions to $u_H^{\text{initial}} = 1.0$ and $u_H^{\text{final}} = 0.8$, respectively, so the exact values depend on q_5 and k for each run. All dimensionful quantities quoted from now on should be understood as expressed in units set by the value of the initial horizon u_H^{initial}.

The condition that the charge had to remain constant only appeared for the case with axial magnetic field, but we fixed its value to $q_5 = 1.0$ for all the cases. The benefit of this is that all the runs have the same initial and final chemical potential and, therefore, they can be compared. However, the initial and final temperatures will not be the same for the different cases when we compare runs with different values of k. The specific value of the charge has no special meaning and only affects the results by a normalization. However, we used this value in order for the flow to have a sufficiently large value that can be clearly seen in the results.

6.6.1 Momentum conservation

The first analysis we decided to make was reminiscent of some of the results in [106]. We wanted to see the impact that the time span of the quench τ had on the equilibration process on both cases without momentum relaxation, by looking at the results for several different values of this parameter.

In Figure 6.1 we show the results for the case with vector magnetic field. It can be seen there are two different regimes. The first one is characterized by overshooting before the equilibration finishes. We call them fast quenches. If the quench is fast enough it also shows what we call delay. By delay we mean that the response in the current builds up mainly after the time dependent perturbation (6.15) has already finished. We will quantify this delay more precisely later. The other regime, slow quenches, show smooth monotonic behavior and have no delay. In the limit of very large τ it approaches the near equilibrium approximation based on (6.4). The transition between both regimes appears at around $\tau \approx 1$. We note that this is significantly simpler than the behavior observed in [106] in the case of the gravitational anomaly induced CME, where three different regimes of fast, intermediate and slow quenches could be distinguished.

In Figure 6.2 we show the results for the case with axial magnetic field. Both regimes can be again observed in this case and we can see that the equilibration times are essentially the same ones. However, the rest of the behavior is different due to the appearance of non vanishing flow, as discussed above. The final equilibrium value is somewhat larger than what could be expected from applying the CME formula alone. We attribute this to the convective flow component present in the final state as outlined in the section 6.5. The fluid velocity in the final state can be computed from eq. (6.14) by demanding that the final and initial energy currents (momentum density) are the same. Once this flow component is taken into account the axial current in the final state matches the expectation from hydrodynamics perfectly (6.13) as can be seen from the black continuous line in Figure 6.2.

In the case of fast quenches, the overshooting can be checked to never cross below the value the current would have if only the anomalous contribution was present.

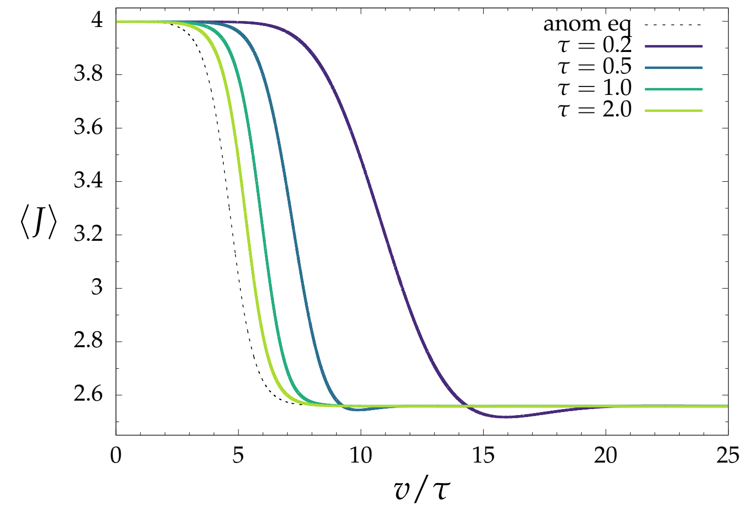

FIGURE 6.1: Out of equilibrium electromagnetic current for different values of the time span of the quench with no momentum relaxation. The black dashed line is the near equilibrium approximation we use as a reference to signal out of equilibrium behavior.

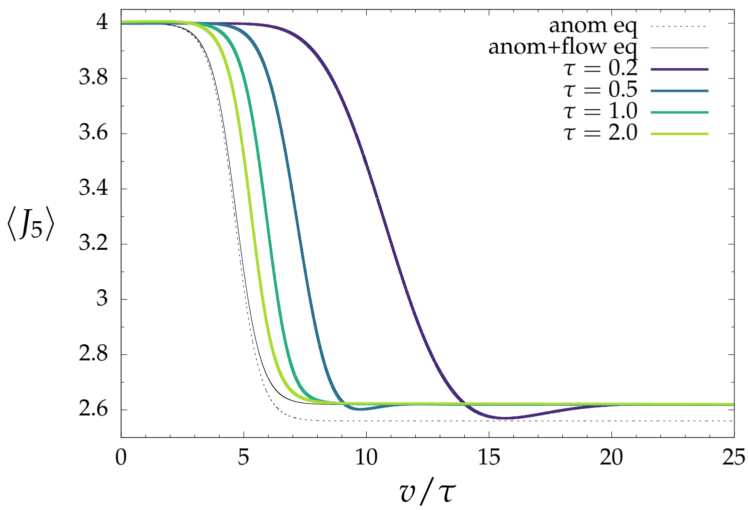

FIGURE 6.2: Out of equilibrium axial current for different values of the time span of the quench. The black line is the near equilibrium approximation we use as a reference to signal out of equilibrium behavior.

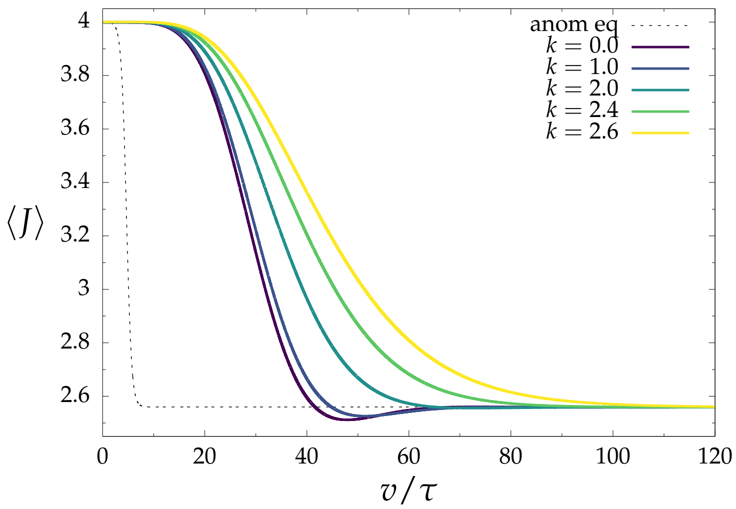

FIGURE 6.3: Out of equilibrium electromagnetic current for different values of the momentum relaxation parameter k. The black dashed line is the near equilibrium approximation we use as a reference to signal out of equilibrium behavior.

6.6.2 Momentum relaxation

We now look at the results with momentum relaxation. In order to do that, we compare the quenches for several different values of k, always keeping $\tau = 0.05$. We start by looking at the case with vector magnetic field. We see again in Figure 6.3 the appearance of two different regimes for the current, although now the transition happens for $k \lesssim 2$.

For the case with axial magnetic field, we obtain Figure 6.4 and 6.5. In the current, it can be observed that there are now two equilibration processes, with different time scales, and their interaction gives a richer structure. One of them is already present for $k = 0$ and is produced by the change in the mass. The other one, only present for $k \neq 0$ is precisely related to the disappearance of flow due to momentum relaxation. For small values of k, like $k = 1.0$ in the plots, this second process is so slow that the current almost equilibrates to the value with full flow before slowly decreasing towards the result with no flow. For bigger values of k, though, the momentum relaxation is faster and the two processes cannot be seen as independent, although now the whole process is much longer than for the case without momentum relaxation.

The plot for the energy current shows a much simpler structure. For no momentum relaxation, the energy current stays constant as it is equal to the momentum density, which is a conserved current. For $k \neq 0$ it interpolates between the initial and final equilibrium values of the anomalous contribution, and the process is faster for bigger k's. However, there is a point, around $k = 2.5$ where this trend changes and the process starts being slower for bigger k. As we will see this is in agreement with the quasi-normal mode structure of the system.

Finally, in order to better understand the structure of the cases with axial magnetic field and momentum relaxation, we now plot the results for different values of τ and a fixed value of $k = 1$ in Figures 6.6 and 6.7. It can be seen that, as in

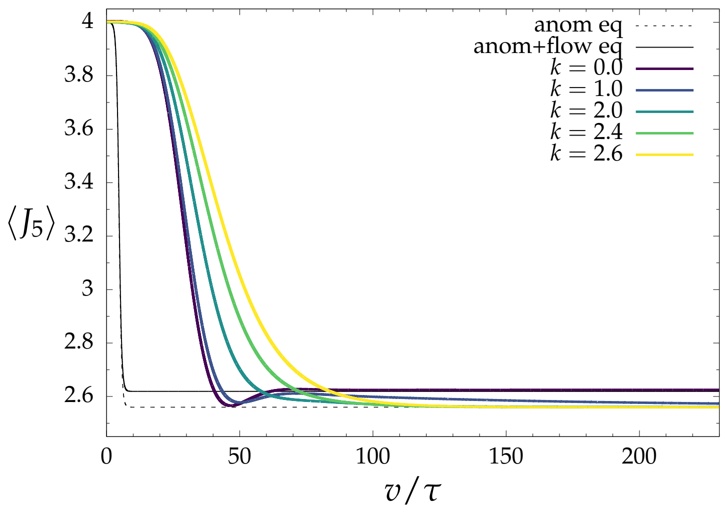

FIGURE 6.4: Out of equilibrium axial current for different values of the momentum relaxation parameter k. The black line is the near equilibrium approximation we use as a reference to signal out of equilibrium behavior.

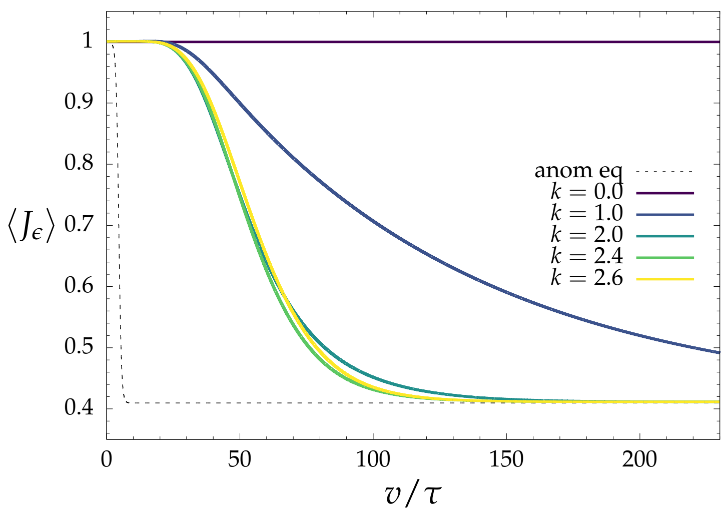

FIGURE 6.5: Out of equilibrium energy current for different values of the momentum relaxation parameter k. The black dashed line is the near equilibrium approximation we use as a reference to signal out of equilibrium behavior.

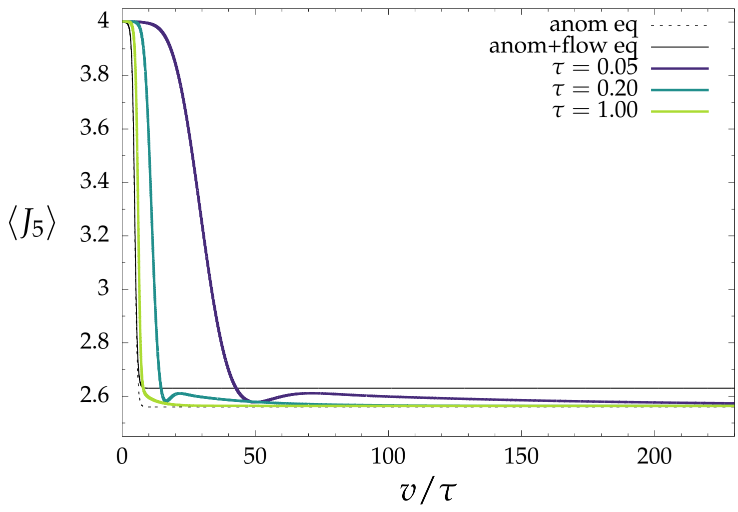

FIGURE 6.6: Out of equilibrium axial current for different values of the time span of the quench with momentum relaxation parameter $k = 1$. The black dashed line is the near equilibrium approximation we use as a reference to signal out of equilibrium behavior.

all previous cases, the results are closer to the near equilibrium approximation as τ becomes larger. However, the structure of the time evolution of the axial current is richer. As τ becomes smaller, it can be seen that momentum relaxation is not dominant enough to fully suppress the appearance of flow. Therefore, the response first tends to equilibrate to the value it would have if $k = 0$, only to subsequently deviate from that value and end up reaching the final equilibrium value through the disappearance of flow.

6.6.3 Delay

One interesting feature in the results obtained is what we call delay. It can be defined as the time lapsed after the quench has finished and before the response in the different currents starts to build up. It appears for fast quenches in both the vector and axial currents. However, we will concentrate on the vector case because it is the one that could potentially have interesting phenomenological implications for heavy ion collisions, as we will comment on below.

In particular, we define our delay in the following way. We considered the quench to be finished when the value of $8\kappa\mu B$, with μ obtained from (6.4), deviates less than 0.1% from the final equilibrium value. In an analogous way, we considered the buildup in the current to start when its value deviates more than 0.1% from the initial value. The delay Δ is equal to the difference in v between those two instants.

We expect the delay to depend on the momentum relaxation coefficient k and the time length of the quench τ. The results are presented in Figure 6.8. In the first set of points, we show the dependence on τ for fixed $k = 0$. In the second set of points, we show the dependence on k for fixed $\tau = 0.05$. It can be seen that the delay becomes bigger for bigger k. However, it becomes smaller for bigger τ. In particular, it can become negative for big τ (roughly around 0.1 for the case with $k = 0$), which means that there is no delay and the current starts to build up before the quench is finished.

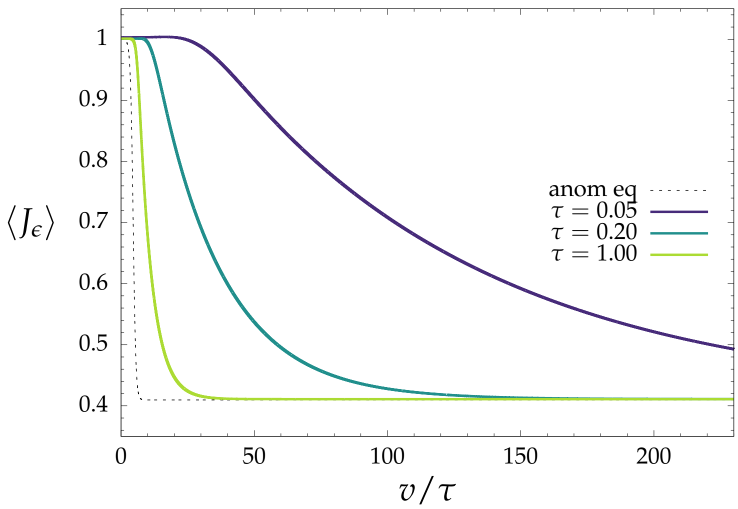

FIGURE 6.7: Out of equilibrium energy current for different values of the time span of the quench with momentum relaxation parameter $k = 1$. The black dashed line is the near equilibrium approximation we use as a reference to signal out of equilibrium behavior.

Thanks to the logarithmic scale, it can be seen that for very fast quenches at fixed k the value gets to a plateau and it has a well-defined finite limit for $\tau \to 0$.

This could have interesting implications for the search of anomalous transport in heavy ion collisions. Our results suggest that the response in the current starts to build up some time after the end of the equilibration process for sufficiently fast quenches. Indeed, equilibration or hydrodynamization is supposed to be fast in heavy ion collisions. At the same time the life-time of the magnetic field is finite. It is also known that the life time of the magnetic field is shorter for higher energy collisions. While it is sometimes assumed that the net effect of stronger magnetic field and shorter lifetime compensate each other our results might point into a different direction. If the life time of the magnetic field is short the delay might mean that no CME current can actually be built up before the magnetic field decays. This could in principle allow the case that the CME signal is suppressed at high energies at the LHC even if it is observable at the lower RHIC energies. Indeed, the current results from experimental searches for CME signals at RHIC and LHC allow precisely such an interpretation [157].

The present model is certainly too simplistic to allow application to a more realistic situation. Our results call however for further detailed studies with more phenomenological input, such as finite lifetime of the magnetic field.

6.7 Quasinormal Modes

Since the equations (6.5)-(6.9) are linear in the fields one might expect that the time evolution can be reasonably well described in terms of quasinormal modes. However, such an analysis is complicated due to the explicit time dependence of the blackening factor. A complete quasinormal mode analyses would therefore have to include also the modes stemming from the non-linear metric equations. Instead

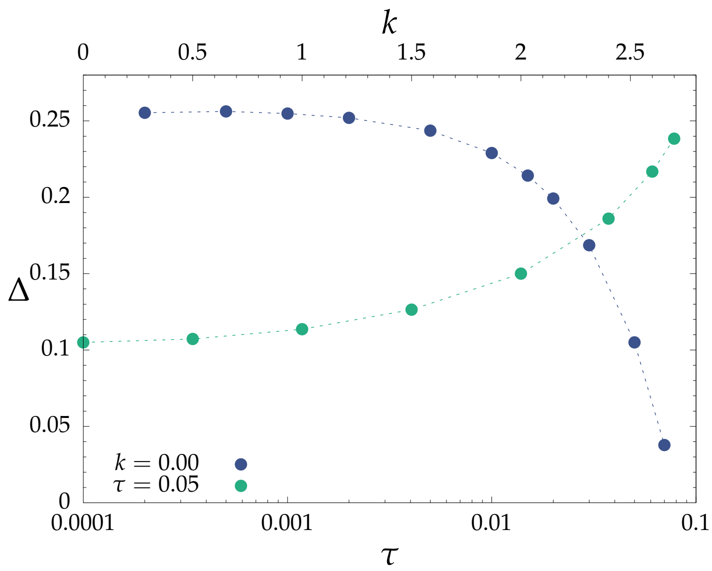

FIGURE 6.8: Delay presented as a function of the length of the quench τ for the case with $k = 0$, and as a function of the momentum relaxation parameter k for the case with $\tau = 0.05$.

we will study a somewhat simpler problem. We consider the final equilibrium state and ask how fast a generic perturbation of that state decays. The equations for the perturbations are the same as the equations (6.5)-(6.9) without the terms with the magnetic field. The equations for the quasinormal modes follow then by writing

$$d = -i\omega - \frac{u^2 f}{2}\partial_u \,.$$

The resulting (system of) equations are solved imposing regularity of the solutions on the horizon and vanishing non-normalizable mode on the boundary. In the case of the system of equations (6.6)-(6.9) one needs to construct three linearly independent solutions. The constraint allows however only for two independent solutions. A trivial third solution can be found by choosing $h = -i\omega$ and $Z = k$.

The proper boundary conditions on the horizon for the gauge fields and the scalar field are that they take finite but non-vanishing values whereas the metric perturbation h has to vanish on the horizon. With this boundary conditions the solutions can be found by numerical integration. In the case of the coupled system of equations the quasinormal modes are found by setting the determinant of the matrix spanned by the three linearly independent solutions to zero at the boundary [5, 93].

We limit ourselves to the dominant mode with largest imaginary part for both cases. The real and imaginary part of the first quasinormal mode of the final equilibrium state are shown as functions of k in Figure 6.9 for the case with vector magnetic field and in Figure 6.10 for the case with axial magnetic field. It can help us understand some of the features of the linear response results. In particular in what refers to the time of final equilibration of the currents, since the background will be equilibrated to this final equilibrium state before the current equilibrates.

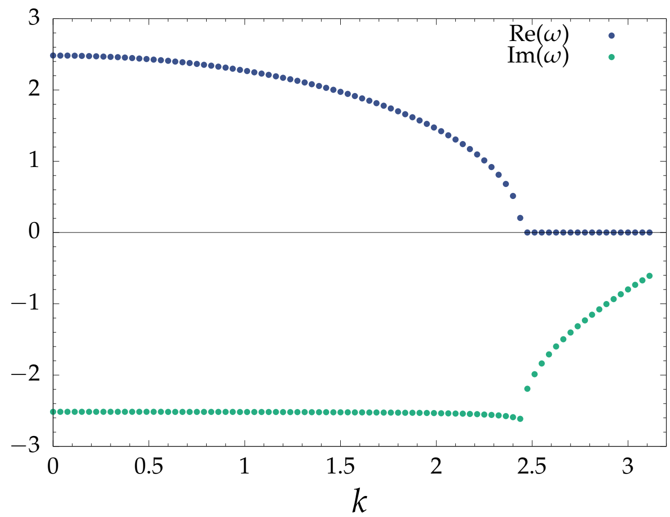

FIGURE 6.9: First quasinormal mode in the final state for the case with vector magnetic field.

For the case with vector magnetic field, the real part decreases while the imaginary part stays the same until the mode becomes purely imaginary slightly below $k = 2.5$. At precisely that point and after increasing a little bit, the absolute value of the imaginary part decreases quite fast. This is in agreement with the equilibration times of Figure 6.3. In that plot the final equilibration seems to happen roughly at the same point for k below 2.4 while for the curves above this value of the momentum relaxation parameter k the equilibration time becomes longer and longer. Since the quasinormal modes come in pairs with opposite sign real part, there are two modes colliding on the imaginary axes. The second mode then moves down the imaginary axes until it pairs up with another pure imaginary mode. In general, a quite complicated pattern of modes moving on and off the imaginary axes develops for higher k values. Since our interest is in the dominating mode we have not further investigated this.

For the case with axial magnetic field, the real part of the dominating mode is always zero and therefore the mode is purely imaginary. One way of understanding this mode is that it is connected to the diffusive mode at $k = 0$. Since the diffusive mode is purely imaginary also this mode has vanishing real part. The absolute value of the imaginary part first increases from zero and, after peaking roughly around $k = 2.5$, it decreases again to approach zero towards extremality, which is slightly above $k = 3.5$. This again shows good qualitative agreement with the energy current results of Figure 6.5. We can see there that the curves approach the final value faster as k grows but after $k = 2.4$ this trend changes and, in fact, the case with $k = 2.6$ equilibrates later.

6.8 Discussion

We have studied out of equilibrium chiral magnetic effect in a holographic setup based on Vaidya-like solutions. An essential and new ingredient was the inclusion

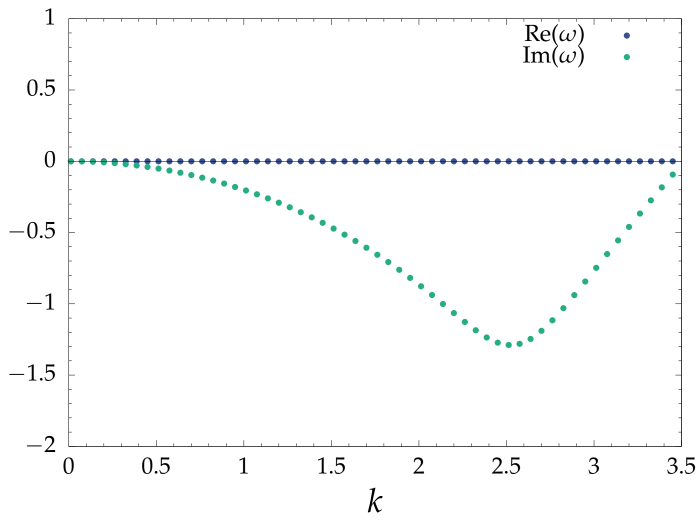

FIGURE 6.10: First quasinormal mode in the final state for the case with axial magnetic field.

of momentum relaxation. Without it the response in the energy momentum tensor would have been trivial.

An interesting feature is that, although the metric and gauge field backgrounds show formally an instantaneous equilibration, the anomaly induced response shows a late equilibration. That proves that far from equilibrium linear response takes place even in Vaidya metrics. In particular, the response is delayed for sufficiently fast quenches. This means that the current needs some time to build up and it even stays at its original equilibrium value for a rather long time. This is qualitatively similar to what has been observed in the previous study [106] based on gravitational anomaly induced chiral transport. Therefore, one might speculate that it is a universal feature of strong coupling out of equilibrium dynamics, or even anomalous transport. If so, it could hold some important lesson for the observation of the CME in heavy ion collisions at RHIC and LHC. While thermalization (or hydrodynamization) is in general believed to be very fast, this might not necessarily mean that the anomaly induced charge separation sets in at the same timescale. There might be significant delay and, since the life time of the magnetic field is smaller at the LHC, this could mean that the chances for observing CME signals at RHIC are larger than those at LHC. One important direction for future research is therefore to develop better holographic out of equilibrium models of anomalous transport taking into account the finite lifetime of the magnetic field.

An interesting side result of our study is that the Vaidya metric allows for straightforward inclusion of momentum relaxation with massless scalar fields in five dimensions. A curious feature is that the response in the non-conserved energy current builds up faster for some intermediate value of the momentum relaxation parameter $k \approx 2.5$ than for smaller or larger values. This seems to be in agreement with the quasinormal mode spectrum. The value of the dominating mode first decreases, finds a minimum at around $k = 2.5$ and then starts to increase again. This indicates that for values $k > 2.5$ the system relaxes slower to equilibrium and we can indeed see this also in the data for the time evolution of the energy current.

Chapter 7

Conclusions

We are finally approaching the end of this thesis. In the previous chapters, different works studying anomalous transport though holography have been presented. We have broaden our understanding of anomaly induced transport by shedding some light on particular aspects of it. We will now use this final chapter to discuss the results obtained in this thesis and their meaning, and we will also comment on some of the open questions and the paths that one might want to explore in the future.

In Chapter 3, we have exploited the existence of gravitational conserved charges and the possibility to find them using Wald's procedure in order to compute the field theory observables in terms of quantities evaluated at the horizon. According to the holographic RG flow, this must be interpreted as a signal that the anomalous response in the charge and energy currents is determined from the low energy physics of the theory. Although some hints about the use of that construction already existed in the literature, as we discuss at the end of Section 3.1, we managed to synthesize those approaches and apply them in the presence of pure gauge and mixed gauge-gravitational anomalies to charge and energy currents.

In Chapter 4, we have found that anomalous transport does not depend in equilibrium on the inclusion of momentum relaxation and disorder in the charged sector. In order to do that, we extended the models already proposed by [14, 59] to a five-dimensional gravitational theory in AdS. An interesting aspect of these results is that they serve as a corollary of Chapter 3, since we study a theory in which the new contributions to the stress tensor proposed in that chapter are nonvanishing. Along the way, we have also computed the electric DC response. The results are very similar to the ones they found for four dimensions. The response can become zero or even negative for some values of the momentum relaxation parameter.

In Chapter 5, we have analyzed the holographic model of Weyl semimetals that was proposed by our research group some months before the beginning of the PhD. This model is inspired by a field theory model and its main feature is a quantum phase transition between a Weyl semimetal phase and a trivial semimetal phase. In the work included in this thesis, we have explored the parameter space of the model and we have found that such a phase transition is universal for a large range of bulk mass and quartic coupling. Besides that, we have extended the computation of the anomalous Hall effect to the axial Hall effect. The expected result from the algebraic structure of the two different anomalies involved could only be recovered after considering a nontrivial renormalization of the external axial fields in the relevant diagrams.

In Chapter 6, we have extended the study of anomalous transport to out of equilibrium setups introducing time dependence through a generalization of Vaidya metrics with momentum relaxation. We have found that the response takes some time to build up and the relevant time scales depend on the velocity of the quench and the momentum relaxation parameter. We propose this as an explanation of why the experimental results of RHIC seem to be more compatible with the chiral magnetic effect than those of LHC. In this picture, the anomalous response would take some time to build up and for very energetic collisions it might be the case that the magnetic field is already switched off by the time the response could be relevant.

All these projects and results leave some space for future work. One of the first questions that might arise from Chapter 3 is what is the meaning of the extrinsic curvature for the dual field theory. This object is proportional to the metric in the boundary and is usually responsible for introducing temperature in the horizon, but we have seen it plays a less obvious role in the rest of the bulk. Therefore, one might wonder what is its dual at intermediate energy scales. Furthermore, it introduces a new operator in the diffeomorphism Ward identity that only vanishes at the boundary. What is the meaning of this new operator? The answer is not evident but we consider this to be a very interesting question.

Another question that arises from the definition of the membrane currents from Chapter 3 is if these definitions are also valid in out of equilibrium situations. It is necessary to include a time-like Killing vector to obtain the diffeomorphism Komar charges that allow to find the radial conservation equations. However, such a Killing vector bounds the result to equilibrium. Therefore, it is not clear what the new terms in the definition of the currents mean for non equilibrium. Analyzing this in detail might also give some hints about the role of the extrinsic curvature for the dual field theory. Of course, the study must be systematic and it should be extended to all the rest of the coefficients. We could check to what extent the behavior seen in Chapter 6 is universal or if, on the contrary, it is specific to the chiral magnetic effect.

In Chapters 4 and 6, we have introduced systems in which momentum is not conserved. While this behavior has no influence on anomalous transport in equilibrium, we have shown that it does have an implication out of equilibrium. However, one question that arises is what would be the effect on anomalous transport of a mechanism that broke energy conservation. Furthermore, would it affect it only out of equilibrium or also in equilibrium? Thus, we consider the search for a mechanism to break energy relaxation in holography very interesting.

In Chapter 6, we have obtained results that we try to use in order to gain some intuition about the experimental results of quark-gluon plasma in heavy ion collisions. However, it is difficult to know how well we can model the quark-gluon plasma with holography. We are actually working on gravitational theories that are at best dual to super Yang-Mills, not to QCD. Thus, more work should be done on trying to understand how valuable these holographic computations really are for the phenomenology of heavy ion collisions. Furthermore, our modeling of the quark-gluon plasma using holography should be improved by choosing realistic values of the physical parameters and introducing magnetic fields with a finite lifetime.

Anisotropy has proved to introduce quite exotic behavior in several setups, like the holographic Weyl semimetal analyzed in Chapter 5. In particular, in this model anisotropy breaks the well-known lower bound on the viscosity to entropy ratio [104] and it also breaks a bound on the butterfly effect velocity of quantum chaos [15]. Therefore, one might wonder if anisotropy can also have some surprising effect out of equilibrium for anomalous transport.

We conclude the thesis with the suggestion of these future projects. The field of anomalous transport has been prolific in the last years, but many things remain to be done. We should not forget that much of the interest of these transport phenomena stems from their hybrid nature: they allow us to study exciting experimental systems and, at the same time, explore some deep physical concepts. Therefore, this has been one of the main leitmotifs of this thesis. To perform theoretical work that helps us grasp the subtleties of anomalous transport and thus contribute to a better understanding of the experimental systems in which it appears. We hope the thesis has been able to achieve this goal.

Conclusiones

Estamos alcanzando el final de la tesis. En los capítulos previos hemos presentado distintos trabajos en los que se utiliza la holografía para estudiar transporte anómalo. Hemos ampliado nuestro entendimiento del transporte inducido por anomalías a través de la elucidación de aspectos particulares de él. Vamos a utilizar este capítulo final para discutir los resultados obtenidos en esta tesis y su significado, así como para plantear algunas de las preguntas que surgen y caminos que se podrían explorar en el futuro.

En el Capítulo 3, hemos utilizado la existencia de cargas conservadas gravitacionales y la posibilidad de encontrarlas usando el procedimiento de Wald para calcular los observables de teoría de campos en términos de cantidades evaluadas en el horizonte. De acuerdo al flujo del grupo de renormalización holográfico, esto debe de ser interpretado como una señal de que la física a bajas energías de la teoría es la encargada de determinar la respuesta anómala de las corrientes de carga y energía. Aunque algunas pistas de que tal construcción se podía utilizar con este objetivo ya existían en la literatura, como discutimos al final de la Sección 3.1, hemos sido capaces de unir los distintos cabos sueltos y aplicarlos en presencia de anomalías quirales puras y mixtas al cálculo de corrientes de carga y energía.

En el Capítulo 4, hemos hallado que el transporte anómalo no depende en equilibrio de la rotura de la conservación del momento o de la inclusión de desorden en el sector cargado. Con este propósito, extendimos los modelos propuestos por [14, 59] a una teoría gravitatoria de cinco dimensiones en AdS. Un aspecto interesante de este resultado es que sirve de colorario al Capítulo 3, ya que estudiamos una teoría en la que las nuevas contribuciones al tensor energía-momento propuestas en aquel capítulo son no nulas. Además, hemos calculado la respuesta eléctrica de corriente continua. Los resultados son muy similares a los de cuatro dimensiones. La respuesta es cero o negativa para ciertos valores del parámetro de relajación del momento.

En el Capítulo 5, hemos analizado un modelo holográfico de semimetales de Weyl que fue propuesto por nuestro grupo de investigación unos meses antes del inicio de este doctorado. Dicho modelo está inspirado en uno de teoría de campos y su principal ingrediente es una transición de fase cuántica entre una fase de semimetal de Weyl y otra fase trivial. En el trabajo incluido en esta tesis hemos explorado el espacio de parámetros del modelo y hemos encontrado que la transición de fase es universal para un gran rango de la masa del *bulk* y del acoplo cuártico. Además, hemos extendido el cálculo del efecto Hall anómalo al efecto Hall axial. Se puede ver que el resultado que cabría esperar a partir de la estructura algebraica de las dos anomalías se recupera una vez consideramos una renormalización no trivial de los campos axiales externos en los diagramas relevantes.

En el Capítulo 6, hemos extendido el estudio de transporte anómalo a sistemas fuera del equilibrio introduciendo dependencia temporal gracias a una generalización de las métricas de Vaidya que tiene relajación de momento. Hemos obtenido que la respuesta empieza a aparecer después de un cierto tiempo y que las escalas de tiempo relevantes dependen de la velocidad del *quench* y del parámetro de relajación de momento. Proponemos este fenómeno como una explicación de por qué los resultados experimentales de RHIC parecen más compatibles con la aparición del efecto quiral magnético que los resultados del LHC. Según nuestro argumento, la respuesta anómala tardaría un tiempo en aparecer y para colisiones muy energéticas podría darse el caso de que el campo magnético fuese despreciable para el momento en el cual la respuesta se hiciese relevante.

Todos estos proyectos y resultados dejan espacio para trabajo futuro. Una de las primeras preguntas que surgen del Capítulo 3 es cuál es el significado de la curvatura extrínseca para la teoría de campos dual. Este objeto es proporcional a la métrica en el borde y habitualmente introduce la temperatura en el horizonte, pero su rol en el resto del bulk no está claro. Por lo tanto, cabe la duda sobre el papel que juega a escalas de energías intermedias. Más aún, introduce un nuevo operador en la identidad de Ward de difeomorfismos que solo se anula en el borde. ¿Qué significa este operador? La respuesta no es evidente pero consideramos que se trata de una pregunta interesante.

Otra duda que surge de la definición de corrientes de membrana del Capítulo 3 es si estas definiciones siguen siendo válidas en situaciones fuera del equilibrio. Se necesita un vector de Killing temporal para obtener las cargas de Komar asociadas a difeomorfismos que permitan encontrar las ecuaciones de conservación radial. Sin embargo, un vector de Killing de esta forma restringe el resultado al equilibrio. Por tanto, no está claro el significado fuera del equilibrio de los nuevos términos en las definiciones de las corrientes. Un análisis detallado podría aportar además algunas pistas sobre el papel de la curvatura extrínseca para la teoría de campos dual. Por supuesto, el estudio debería ser sistemático y debería ser extendido a todos los demás coeficientes. De esta manera, podríamos comprobar hasta qué punto el comportamiento del Capítulo 6 es universal o, por el contrario, específico del efecto quiral magnético.

En los Capítulos 4 y 6 hemos introducido sistemas en los que el momento no es conservado. Mientras que este comportamiento no tiene influencia en el transporte anómalo en el equilibrio, hemos mostrado que sí que tiene consecuencias en la respuesta fuera del equilibrio. Sin embargo, cabe pensar qué efecto podría tener sobre el transporte anómalo un mecanismo que rompiese la conservación de la energía y si afectaría solo fuera del equilibrio o también en equilibrio. Por tanto, consideramos que sería muy interesante buscar la manera de introducir la rotura de la conservación de la energía en holografía.

En el Capítulo 6, hemos obtenido resultados que intentamos utilizar para ganar intuición sobre los resultados experimentales del plasma de quarks y gluones en colisiones de iones pesados. Sin embargo, es difícil saber cómo de bien podemos modelizar este plasma con la holografía. Trabajamos en teorías gravitatorias que en el mejor de los casos son duales a super Yang-Mills, y no a QCD. Por consiguiente, es importante continuar el trabajo para elucidar cómo de valiosos son estos cálculos holográficos para la fenomenología de las colisiones de iones pesados. Más aún, nuestro modelizado holográfico del plasma de quarks y gluones debería ser mejorado escogiendo valores realistas para los parámetros físicos e introduciendo campos magnéticos con vida finita.

Se sabe que la anisotropía es la responsable de introducir comportamientos exóticos en varios sistemas, como el semimetal de Weyl holográfico analizado en el Capítulo 5. En particular, en este modelo la anisotropía rompe la famosa cota del ratio de viscosidad frente a entropía [104] y la cota de la velocidad del efecto mariposa de caos cuántico [15]. Por tanto, tiene sentido preguntarse también si la anisotropía podría tener algún efecto sorprendente en el transporte anómalo fuera del equilibrio.

Concluimos esta tesis con la sugerencia de estos proyectos futuros. El campo del transporte anómalo ha sido prolífico en los últimos años, pero todavía quedan muchas cosas por hacer. No debemos olvidar, además, que gran parte del interés de estos fenómenos de transporte se debe a su naturaleza híbrida: nos permiten estudiar excitantes sistemas experimentales y, al mismo tiempo, explorar conceptos físicos muy profundos. Por lo tanto, este ha sido uno de los principales *leitmotiv* de esta tesis. Realizar un trabajo teórico que nos ayude a entender las sutilezas del transporte anómalo y así contribuir a una mejor comprensión de los sistemas experimentales en los que aparece. Esperamos que la tesis haya sido capaz de cumplir con este objetivo.

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